Core I Algebra Exam

August, 2001

Instructions: Do problems 1* and 2* and any two others for a total of 4 problems. Put your name on the top right corner of each sheet, and write on only one side of each sheet. Copy each problem before solving it. Turn in ONLY the four problems you selected. You have 2 and 1/2 hours for this test. Good luck!

1*

(a). State the structure theorem for finitely generated abelian groups.

(b). If p and q are distinct primes determine the number of nonisomorphic abelian groups of order p^3q^4. Give an explanation not just a numerical answer.

2*

(a). Let F be a field and let A be an $n \times n$ matrix with entries in F. State a necessary and sufficient condition on the minimal polynomial of A for A to be diagonalizable over F.

(b). Let $F = C$ be the field of complex numbers. If A satisfies the equation $A^3 = -A$, then show that A is diagonalizable over C.

(c). Let $F = R$ be the field of real numbers. Given that A satisfies the equation $A^3 = -A$, and given that A is diagonalizable over R, what is the strongest conclusion that can be drawn about A?

3. Show that the set of all elements of finite order in an abelian group form a subgroup.

4. Let R be a finite integral domain (commutative, with 1). Prove that R is a field.

5.

(a). Let R be a ring and M an R-module. What does it mean for M to be a *free* R-module?

(b). Let $Z\left[\frac{1}{2}\right]$ denote the subring of Q generated by Z and $\frac{1}{2}$. Prove or disprove: $Z\left[\frac{1}{2}\right]$ is a free Z-module.

6. Let F be a field. Construct, up to similarity, all linear transformations $T : F^6 \longrightarrow F^6$ with minimal polynomial $m_T(X) = (X-5)^2(X-6)^2$.