CORE II ALGEBRA
August, 2000

Directions: You must answer the three starred problems. Then, answer three of the remaining five problems. Let $\mathbf{Q}$ denote the field of rational numbers.

(*) 1. Let K denote the splitting field over $\mathbf{Q}$ of the polynomial $X^3 - 2$. Determine the Galois group of K over $\mathbf{Q}$, all intermediate subfields of this extension, and set up the Galois correspondence between intermediate fields and subgroups of the Galois group.

(*) 2. (a) Construct a field with precisely 27 elements.
(b) Classify the additive and multiplicative groups of this field (according to the Fundamental Theorem of Finite Abelian Groups).
(c) Determine the Galois group of this field over the field with 3 elements.

(*) 3. Prove that every group of order 15 must be cyclic. (Note: You should not assume the group is abelian.)

4. (a) How many 3-Sylow subgroups may exist in a group of order 12?
(b) Give examples to show that each possibility in (a) actually occurs.

5. Prove Cayley's Theorem: Every finite group is isomorphic to a subgroup of a group of permutations.

6. Let $f(x)$ be an irreducible polynomial of degree 6 over a field F. Let K be an extension field of F with $[K : F] = 2$. Prove that if $f(x)$ is not irreducible over K, then it factors in $K[x]$ into the product of two irreducible cubic polynomials.

7. Let G be a nonabelian group of order p^3, where p is a prime. What is the order of the center of G? (Justify your answer.)

8. Let K be a Galois extension of $\mathbf{Q}$ such that $G(K/\mathbf{Q})$ is a cyclic group of order 30.
(a) How many intermediate fields are there and what are their degrees over $\mathbf{Q}$?
(b) Give an example of such an extension K. (Hint: 31 is a prime.)