CORE II ALGEBRA

August 2001

Directions: There are five problems. Solve each one on a separate sheet of paper, with your name and the problem number at the top. You have 3 and 1/2 hours. Good Luck!

1. Prove for the field extension $F$ over $K$ of finite degree: The order of the Galois group $\mathrm{Gal}(F/K)$ is equal to the degree of $F$ over $K$ if and only if the fixed field of $\mathrm{Gal}(F/K)$ is equal to $K$.

2. Let $p$ be an odd prime number. Give all the definitions and prove: The regular $p$-gon is constructible by ruler and compass if and only if $p$ is a Fermat prime.

3. Let $\mathbf{Q}$ be the field of rational numbers and let $z$ denote a primitive 9th root of unity. Identify all intermediate fields of $\mathbf{Q}(z)$ over $\mathbf{Q}$.

4. Let $\mathbf{Q}$ be the field of rational numbers and let $z$ denote a primitive 32nd root of unity. Put $F = \mathbf{Q}(z)$. Illustrate by a diagram the 1:1 correspondence between intermediate fields of $F$ over $\mathbf{Q}$ and subgroups of the Galois group $\mathrm{Gal}(F/\mathbf{Q})$.

5. Let $K$ be a field and $f$ an irreducible polynomial in one indeterminate with coefficients in $K$. Prove that $f$ need not be separable.