CORE II ALGEBRA
January, 2001

Directions: You must answer the three starred problems. Then, answer three of the remaining five problems. Let $\mathbf{Q}$ denote the field of rational numbers.

(*) 1. Let K, E, and F be fields such that $K \subset E \subset F$.
(i) Prove that if F is algebraic over E and E is algebraic over K, then F is algebraic over K.
(ii) Give an example of fields K, E, and F such that F is Galois over E and E is Galois over K, but F is not Galois over K.

(*) 2. (i) Construct a field with precisely 16 elements.
(ii) Does your field in (i) contain a subfield with precisely 8 elements? If so, tell what it is; if not, tell why not.
(iii) Determine the Galois group of the field in (i) over the field with 2 elements.

(*) 3. Prove that there is no simple group of order 56.

4. (i) If $F = \mathbf{Q}(\sqrt{2}, \sqrt{3})$, find $[F : \mathbf{Q}]$ and a basis of F over $\mathbf{Q}$.
(ii) Do the same for $F = \mathbf{Q}(i, \sqrt{3}, \zeta)$, where i is a complex number whose square is -1 and ζ is a primitive (complex) cube root of 1.

5. Prove Cayley's Theorem: Every finite group is isomorphic to a subgroup of a group of permutations.

6. Let p be a prime. Let G be a group of order $p^e r$, where p does not divide r and e is a positive integer. Suppose H is a normal subgroup of G of order $p^e s$. Prove that every Sylow p-subgroup of G is contained in H.

7. Let G be a group of order 8.
(i) If $g \in G$, show that the conjugacy class of g cannot consist of precisely 4 elements. (Hint: If that did happen, then what would that say about the centralizer of g?)
(ii) What are the possible class equations for a group of order 8? Justify your answer.

8. Let K be a finite Galois extension of F and put $G = G(K/F)$. For $\alpha \in K$, define the trace $\mathrm{Tr}(\alpha)$ and the norm $\mathrm{N}(\alpha)$ by

$$\mathrm{Tr}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha) \quad \text{and} \quad \mathrm{N}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha).$$

Prove that $\mathrm{Tr}(\alpha)$ and $\mathrm{N}(\alpha)$ lie in F.