Algebra Exam Syllabus

The Algebra comprehensive exam covers four broad areas of algebra: (1) Groups; (2) Rings; (3) Modules; and (4) Linear Algebra. These topics are all covered in the first semester graduate algebra course Math 7200, although the more elementary portions of linear algebra are covered in any undergraduate linear algebra course, e.g., Math 4153. Questions from all four areas can be expected, with the major portion of the questions coming from the 7000-level material.

Naturally, not every topic in the following detailed list will be on every exam. However, all listed topics provide a potential source of questions. Examples of representative questions follow the syllabus. The exam itself will consist of five questions (some exams may require one to do five out of six questions) similar to the sample questions given below. In fact, many of these questions were extracted from one (or more) past exams. You will have two and one-half hours to complete the exam.

Hint: In preparing for the exams, a candidate should become familiar with examples and counterexamples, and should develop a facility for computing with specific examples.

The following standard textbooks in algebra have been used in this department in recent years. References to the relevant chapters of each are given in the detailed list of topics which follows. These texts are intended only as examples where one may find the topics tested; the same topics may be found in many other books. Moreover, there is substantial overlap in the texts. In fact, each of the texts contains most, if not all, of the topics listed, plus many other topics.

Reference Texts

[A] M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, NJ, 1991.
[AW] W. A. Adkins and S. H. Weintraub, *Algebra: An Approach via Module Theory*, Graduate Texts in Math. 136, Springer-Verlag, New York, 1992.
[H] T. W. Hungerford, *Algebra*, Graduate Texts in Math. 73, Springer-Verlag, New York, 1974.

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1. Group Theory

Groups, subgroups, normal subgroups, cosets, quotient groups, Lagrange's theorem, homomorphisms, Noether isomorphism theorems, abelian groups, center of a group, commutator subgroup, direct products, fundamental theorem of finitely generated abelian groups as an application of the structure theorem for finitely generated modules over a Euclidean Domain (See §3), Examples: cyclic groups, simple groups, S_n (symmetric group), A_n (alternating group), dihedral groups, matrix groups, i.e., $\mathrm{GL}(n, k)$ and its subgroups.

References: [A], Chapter 2; [AW], Chapter 1; [H], Chapter 1, §1–6, 8, Chapter 2, §1, 2.

2. Rings

Rings, subrings, homomorphisms, ideals, quotient rings, Noether isomorphism theorems for rings, prime ideals, maximal ideals, units, divisibility, polynomial rings, principal ideal domains (PID's) and Euclidean domains: especially $\mathbb{Z}$ and $k[X]$ for k a field, Euclidean algorithm and computation of greatest common divisor, unique factorization in Euclidean domains and polynomial rings over Euclidean domains, Gauss's lemma, field of quotients of an integral domain, fields, construction of finite fields.

References: [A], Chapter 10, Chapter 11, §1–4; [AW], Chapter 2; [H], Chapter 3.

3. Modules

Modules, submodules, homomorphisms, quotient modules, Noether isomorphism theorems for modules, direct sums and direct products, exact sequences, free modules, torsion modules, structure of finitely generated modules over a Euclidean domain, especially over $\mathbb{Z}$ and $k[X]$: annihilator ideals, invariant factors, elementary divisors, application to Jordan and rational canonical form for linear operators (see §4), application to finitely generated abelian groups (see §1), diagonalization of integer matrices and application to generators and relations for finitely generated abelian groups.

References: [A], Chapter 12, [AW], Chapter 3, §4.4, §5.1–5.4; [H], Chapter 4.

4. Linear Algebra

Vector spaces, subspaces, quotient spaces, dual spaces, Hom (V, W). Basis, dimension, linear transformations, matrices, change of basis, determinants. Theory of a single linear operator: eigenvalues, eigenvectors, generalized eigenvectors, similarity, diagonalization and triangularization of operators, functions of matrices and operators, characteristic and minimum polynomials, Jordan canonical form, rational canonical form. (The theory of a single linear transformation can (and probably should) be viewed as a special case of the structure theorem for finitely generated $k[X]$-modules. See §3.) Inner product spaces, Hermitian, unitary, orthogonal and normal operators, adjoints, spectral theorem for normal operators, bilinear and quadratic forms, diagonalization of quadratic forms, signature and Sylvester's law for real quadratic forms.

References: [A], Chapters 3, 4, 7, and 12; [AW], Chapters 4, 5, 6; [H], Chapter 7; Hoffman–Kunze, *Linear Algebra* Chapters 6 and 7, also has the theory of a single linear transformation, Chapters 8, 9, 10 has the theory of inner products spaces and bilinear forms.

Sample Algebra Questions

The following questions are representative of the type and difficulty of the questions which can be expected on the comprehensive exam. For convenience, the problems are divided (very roughly) into the four divisions listed under the syllabus. Some questions could just as easily be included in more than one subdivision. Notation which will be standardized for all exercises is the following: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will denote respectively, the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers. The integers modulo n will be denoted $\mathbb{Z}/n\mathbb{Z}$, while $\mathbb{F}_q$ will refer to a finite field with q elements.

Group Theory

G1. Let H be a normal subgroup of a group G, and let K be a subgroup of H.
  (a) Give an example of this situation where K is not a normal subgroup of G.
  (b) Prove that if the normal subgroup H is cyclic, then K is normal in G.

G2. Prove that every finite group of order at least three has a nontrivial automorphism.

G3. (a) State the structure theorem for finitely generated abelian groups.
  (b) If p and q are distinct primes determine the number of nonisomorphic abelian groups of order p^3q^4.

G4. Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is a prime.
  (a) Show that G has order $(p^2 - 1)(p^2 - p)$.
  (b) Show that for $p = 2$ the group G is isomorphic to the symmetric group S_3.

G5. Let G be the group of units of the ring $\mathbb{Z}/247\mathbb{Z}$.
  (a) Determine the order of G (note that $247 = 13 \cdot 19$).
  (b) Determine the structure of G (as in the classification theorem for finitely generated abelian groups). *Hint:* Use the Chinese Remainder Theorem.

G6. Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. Let $D \subseteq G$ be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbb{R}$ is arbitrary.
  (a) Show that U is a normal subgroup of G and that G/U is isomorphic to D.
  (b) True or False (with justification): $G \cong U \times D$

G7. Let G be a group and let Z denote the center of G.
  (a) Show that Z is a normal subgroup of G.
  (b) Show that if G/Z is cyclic, then G must be abelian.
  (c) Let D_6 be the dihedral group of order 6. Find the center of D_6.

G8. List all abelian groups of order 8 up to isomorphism. Identify which group on your list is isomorphic to each of the following groups of order 8. Justify your answer.

(a) $(\mathbb{Z}/15\mathbb{Z})^*$ = the group of units of the ring $\mathbb{Z}/15\mathbb{Z}$.
(b) The roots of the equation $z^8 - 1 = 0$ in $\mathbb{C}$.
(c) $\mathbb{F}_8^+$ = the additive group of the field $\mathbb{F}_8$ with eight elements.

G9. Let S_9 denote the symmetric group on 9 elements.
(a) Find an element of S_9 of order 20.
(b) Show that there is no element of S_9 of order 18.

G10. $G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$ and $N = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} : c \in \mathbb{R} \right\}$ are groups under matrix multiplication.
(a) Show that N is a normal subgroup of G and that G/N is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}^+$.
(a) Find a group N' with $N \subseteq N' \subseteq G$, with both inclusions proper, or prove that no such N' exists.

G11. Let R be a commutative ring with identity, and let H be a subgroup of the group of units R^* of R. Let $N = \{A \in \mathrm{GL}(n, R) : \det A \in H\}$. Prove that N is a normal subgroup of $\mathrm{GL}(n, R)$ and $\mathrm{GL}(n, R)/N \cong R^*/H$.

G12. Let G be a group of order $2p$ where p is an odd prime. If G has a normal subgroup of order 2, show that G is cyclic.

G13. Prove that every finitely generated subgroup of the additive group of rational numbers is cyclic.

G14. Prove that any finite group of order n is isomorphic to a subgroup of the orthogonal group $\mathrm{O}(n, \mathbb{R})$.

G15. Prove that the group $\mathrm{GL}(2, \mathbb{R})$ has cyclic subgroups of all orders $n \in \mathbb{N}$. (*Hint:* The set of matrices $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are arbitrary real numbers, is a subring of the ring of 2×2 matrices which is isomorphic to $\mathbb{C}$.)

G16. Let H_1 be the subgroup of $\mathbb{Z}^2$ generated by $\{(1, 3), (1, 7)\}$ and let H_2 be the subgroup of $\mathbb{Z}^2$ generated by $\{(2, 4), (2, 6)\}$. Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?

G17. Let H and N be subgroups of a group G with N normal. Prove that $HN = NH$ and that this set is a subgroup of G.

G18. Let $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/30\mathbb{Z}$ and let $H = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/20\mathbb{Z}$. Express the abelian group $\mathrm{Hom}(G, H)$ of homomorphisms from G to H as a direct sum of cyclic groups.

G19. Let G be an abelian group generated by x, y, z subject to the relations

$$\begin{aligned} 15x + 3y &= 0 \\ 3x + 7y + 4z &= 0 \\ 18x + 14y + 8z &= 0. \end{aligned}$$

(a) Write G as a product of two cyclic groups.
(b) Write G as a direct product of cyclic groups of prime power order.
(c) How many elements of G have order 2?

G20. Let $\mathbb{F}$ be a field and let

$$H(\mathbb{F}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F} \right\}.$$

(a) Verify that $H(\mathbb{F})$ is a nonabelian subgroup of $\mathrm{GL}(3, \mathbb{F})$.
(b) If $|\mathbb{F}| = q$, what is $|H(\mathbb{F})|$?
(c) Find the order of all elements of $H(\mathbb{Z}/2\mathbb{Z})$.
(d) Verify that $H(\mathbb{Z}/2\mathbb{Z}) \cong D_8$, the dihedral group of order 8.

G21. Let R be an integral domain and let G be a finite subgroup of R^*, the group of units of R. Prove that G is cyclic.

G22. Let α and β be conjugate elements of the symmetric group S_n. Suppose that α fixes at least two symbols. Prove that α and β are conjugate via an element γ of the alternating group A_n.

G23. Are $(1\,3)(2\,5)$ and $(1\,2)(4\,5)$ conjugate in S_5? If you say "yes", find an element giving the conjugation; if you say "no", prove your answer.

G24. (a) Suppose that G is a group and $a, b \in G$ are elements such that the order of a is m and the order of b is n. If $ab = ba$ and if m and n are relatively prime, show that the order of ab is mn.

(b) Prove that an abelian group of order pq, where p and q are distinct primes, must be cyclic.
(c) If m and n are relatively prime, must a group of order mn be cyclic? Justify your answer.

G25. Let $\varphi : G \to H$ be a *surjective* group homomorphism and let N be a normal subgroup of G. Show that $\varphi(N)$ is a normal subgroup of H. What happens if φ is not surjective? Explain your answer.

G26. Let $Q = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group and $N = \{1, -1, i, -i\}$. Show that N is a normal subgroup of Q. Describe the quotient group Q/N.

G27. Let G be a finite abelian group of odd order. If $\varphi : G \to G$ is defined by $\varphi(a) = a^2$ for all $a \in G$, show that φ is an isomorphism. Generalize this result.

G28. Prove that the direct product of two infinite cyclic groups is not cyclic.

G29. Prove that if a group has exactly one element of order two, then that element is in the center of the group.

G30. Prove that a group of order 30 can have at most 7 subgroups of order 5.

G31. Let $H = \{1, -1, i, -i\}$ be the subgroup of the multiplicative group $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ consisting of the fourth roots of unity. Describe the cosets of H in G, and show that the quotient G/H is isomorphic to G.

G32. (a) Show that the set of all elements of finite order in an abelian group form a subgroup.
(b) Let $G = \mathbb{R}/\mathbb{Z}$. Show that the set of elements of G of finite order is the subgroup $\mathbb{Q}/\mathbb{Z}$.

Ring Theory

R1. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.
(a) Why is R an integral domain?
(b) What are the units in R?
(c) Is the element 2 irreducible in R?
(d) If $x, y \in R$, and 2 divides xy, does it follow that 2 divides either x or y? Justify your answer.

R2. (a) Give an example of an integral domain with exactly 9 elements.
(b) Is there an integral domain with exactly 10 elements? Justify your answer.

R3. Let
$$F = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a, b \in \mathbb{Q} \right\}.$$
(a) Prove that F is a field under the usual matrix operations of addition and multiplication.
(b) Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{2})$.

R4. Let $\mathbb{F}$ be a field and let $R = \mathbb{F}[X, Y]$ be the ring of polynomials in X and Y with coefficients from $\mathbb{F}$.
(a) Show that $M = \langle X + 1, Y - 2\rangle$ is a maximal ideal of R.
(b) Show that $P = \langle X + Y + 1\rangle$ is a prime ideal of R.
(c) Is P a maximal ideal of R? Justify your answer.

R5. Let R be an integral domain containing a field k as a subring. Suppose that R is a finite-dimensional vector space over k, with scalar multiplication being the multiplication in R. Prove that R is a field.

R6. Let R be a commutative ring with identity and let I and J be ideals of R.
(a) Define
$$(I : J) = \{r \in R : rx \in I \text{ for all } x \in J\}.$$
Show that $(I : J)$ is an ideal of R containing I.
(b) Show that if P is a prime ideal of R and $x \notin P$, then $(P : \langle x\rangle) = P$, where $\langle x\rangle$ denotes the principal ideal generated by x.

R7. Let R be a commutative ring with identity, and let I and J be ideals of R.
(a) Define what is meant by the sum $I + J$ and the product IJ of the ideals I and J.
(b) If I and J are distinct maximal ideals, show that $I + J = R$ and $I \cap J = IJ$.

R8. Let $\mathbb{F}_2$ be the field with 2 elements.
(a) Show that $f(X) = X^3 + X^2 + 1$ and $g(X) = X^3 + X + 1$ are the only irreducible polynomials of degree 3 in $\mathbb{F}_2[X]$.
(b) Give an explicit field isomorphism
$$\mathbb{F}_2[X]/\langle f(X)\rangle \cong \mathbb{F}_2[X]/\langle g(X)\rangle.$$

R9. Show that $\mathbb{Z}[i]/\langle 1+i\rangle$ is isomorphic to the field $\mathbb{F}_2$ with 2 elements. As usual, i denotes the complex number $\sqrt{-1}$ and $\langle 1+i\rangle$ denotes the principal ideal of $\mathbb{Z}[i]$ generated by $1+i$.

R10. Consider the ring $\mathbb{Z}[X]$ of polynomials in one variable X with coefficients in $\mathbb{Z}$.
(a) Find all the units of $\mathbb{Z}[X]$.
(b) Describe an easy way to recognize the elements of the ideal I of $\mathbb{Z}[X]$ generated by 2 and X.
(c) Find a prime ideal of $\mathbb{Z}[X]$ that is not maximal.

R11. Determine, with justification, all of the irreducible polynomials of degree 4 over the field $\mathbb{F}_2$ of two elements.

R12. Let $R = \mathbb{Z}[\sqrt{-10}]$.
(a) Show that R is not a PID. (*Hint:* Show that 10 admits two essentially different factorizations into irreducible elements of R.)
(b) Let $P = \langle 7, 5+\sqrt{-10}\rangle$. Show that R/P is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

R13. Suppose that R is an integral domain and X is an indeterminate.
(a) Prove that if R is a field, then the polynomial ring $R[X]$ is a PID (principal ideal domain).
(b) Show, conversely, that if $R[X]$ is a PID, then R is a field.

R14. (a) Prove that every Euclidean domain is a principal ideal domain (PID).
(b) Give an example of a unique factorization domain that is not a PID and justify your answer.

R15. (a) Show that for each natural number $n \in \mathbb{N}$, there is an irreducible polynomial $P_n(X) \in \mathbb{Q}[X]$ of degree n.
(b) Is this true when $\mathbb{Q}$ is replaced by $\mathbb{R}$? Explain.

R16. Prove that the multiplicative group of units $U_5 = ((\mathbb{Z}/5\mathbb{Z})^*, \times)$ is isomorphic to the additive group $(\mathbb{Z}/4\mathbb{Z}, +)$.

R17. Let R be a commutative ring with identity. If $I \subseteq R$ is an ideal, then the radical of I, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{a \in R : a^n \in I \text{ for some positive integer } n\}.$$

(a) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(b) If P is a prime ideal of R and $r \in \mathbb{N}$, find $\sqrt{P^r}$ and justify your answer.
(c) Find $\sqrt{I}$, where I is the ideal $\langle 108\rangle$ in the ring $\mathbb{Z}$ of integers.

R18. (a) Show that $\mathbb{Z}[i]/\langle 3+i\rangle \cong \mathbb{Z}/10\mathbb{Z}$, where i is the usual complex number $\sqrt{-1}$.
(b) Is $\langle 3+i\rangle$ a maximal ideal of $\mathbb{Z}[i]$? Give a reason for your answer.

R19. Let $R = \mathbb{Z}[X]$. Answer the following questions about the ring R. You may quote an appropriate theorem, provide a counterexample, or give a short proof to justify your answer.
(a) Is R a unique factorization domain?
(b) Is R a principal ideal domain?
(c) Find the group of units of R.
(d) Find a prime ideal of R which is not maximal.
(e) Find a maximal ideal of R.

R20. An element a in a ring R is *nilpotent* if $a^n = 0$ for some natural number n.
(a) If R is a commutative ring with identity, show that the set of nilpotent elements forms an ideal.
(b) Describe all of the nilpotent elements in the ring $\mathbb{C}[X]/\langle f(X)\rangle$, where

$$f(X) = (X-1)(X^2-1)(X^3-1).$$

(c) Show that part (a) need not be true if R is not commutative. (*Hint:* Try a matrix ring.)

R21. Let R be a ring, let R^* be the set of units of R, and let $M = R \setminus R^*$. If M is an ideal, prove that M is a maximal ideal and that moreover it is the only maximal ideal of R.

R22. (a) Let R be a PID and let I, J be nonzero ideals of R. Show that $IJ = I \cap J$ if and only if $I + J = R$.
(b) Show that $\mathbb{Z}/900\mathbb{Z}$ is isomorphic to $\mathbb{Z}/100\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ as *rings*.

R23. Let $I = \langle X^2+2,\, 5\rangle \subseteq \mathbb{Z}[X]$ and let $J = \langle X^2+2,\, 3\rangle$. Show that I is a maximal ideal, but J is not a maximal ideal.

R24. Let F be a subfield of a field K and let $f(X),\, g(X) \in F[X] \setminus \{0\}$. Prove that the greatest common divisor of $f(X)$ and $g(X)$ in $F[X]$ is the same as the greatest common divisor taken in $K[X]$.

R25. Find the greatest common divisor of $X^3 - 6X^2 + X + 4$ and $X^5 - 6X + 1$ in $\mathbb{Q}[X]$.

R26. Define $\varphi : \mathbb{C}[X, Y] \to \mathbb{C}[T]$ by $\varphi(X) = T^2$, $\varphi(Y) = T^3$.
  (a) Show that $\mathrm{Ker}(\varphi) = \langle Y^2 - X^3 \rangle$.
  (b) Find the image of φ.

R27. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

R28. Let m, n be two non-zero integers. Prove that the greatest common divisor of m and n in $\mathbb{Z}$ is the same as the greatest common divisor taken in $\mathbb{Z}[i]$. Generalize this to a statement about the greatest common divisor of elements a and b in a Euclidean domain R which is a subring of a Euclidean domain S.

R29. Prove that the center of the matrix ring $M_n(\mathbb{R})$ is the set of scalar matrices, i.e., $C(M_n(\mathbb{R})) = \{aI_n : a \in \mathbb{R}\}$.

R30. Let $R_1 = \mathbb{F}_p[X]/\langle X^2 - 2\rangle$ and $R_2 = \mathbb{F}_p[X]/\langle X^2 - 3\rangle$ where $\mathbb{F}_p$ is the field of p elements, p a prime. Determine if R_1 is isomorphic to R_2 in each of the cases $p = 2$, $p = 5$, and $p = 11$.

R31. (a) Show that the only automorphism of the field $\mathbb{R}$ of real numbers is the identity.
  (b) Show that any automorphism of the field $\mathbb{C}$ of complex numbers which fixes $\mathbb{R}$ is either the identity or complex conjugation.

R32. (a) Find all ideals of the ring $\mathbb{Z}/24\mathbb{Z}$.
  (b) Find all ideals of the ring $\mathbb{Q}[X]/\langle X^2 + 2X - 2\rangle$.

R33. Let R be an integral domain. Show that the group of units of the polynomial ring $R[X]$ is equal to the group of units of the ground ring R.

R34. Express the polynomial $X^4 - 2X^2 - 3$ as a product of irreducible polynomials over each of the following fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_5$.

R35. Let $\omega = (1 + \sqrt{-3})/2 \in \mathbb{C}$ and let $R = \{a + b\omega : a, b \in \mathbb{Z}\}$.
  (a) Show that R is a subring of $\mathbb{C}$.
  (b) Show that R is a Euclidean domain with respect to the norm function $N(z) = z\overline{z}$, where, as usual, $\overline{z}$ denotes the complex conjugate of z.

R36. Let I be an ideal of $\mathbb{R}[X]$ generated by an irreducible polynomial of degree 2. Show that $\mathbb{R}[X]/I$ is isomorphic to the field $\mathbb{C}$.

R37. Show that in the ring M of 2×2 real matrices (with the usual sum and multiplication of matrices), the only 2-sided ideals are $\langle 0 \rangle$ and the whole ring M.

R38. Let R be a commutative ring with identity. Suppose $a \in R$ is a unit and $b \in R$ is nilpotent. Show that $a + b$ is a unit.

R39. (a) Define *prime ideal* and *maximal ideal* in a commutative ring R with identity.
  (b) Let R and S be commutative rings with identities 1_R and 1_S, respectively, let $f : R \to S$ be a ring homomorphism such that $f(1_R) = 1_S$. If P is a prime ideal of S show that $f^{-1}(P)$ is a prime ideal of R.
  (c) Let f be as in part (b). If M is a maximal ideal of S, is $f^{-1}(M)$ a maximal ideal of R? Prove that it is or give a counterexample.

Module Theory

M1. Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Is $\mathbb{Z}\left[\frac{1}{2}\right]$ finitely generated as a $\mathbb{Z}$-module? Justify your answer.

M2. (a) Let R be a ring and M an R-module. What does it mean for M to be a *free* R-module?
  (b) Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}\left[\frac{1}{2}\right]$ is a free $\mathbb{Z}$-module.

M3. (a) Show that $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module.
  (b) Is $\mathbb{Q}$ a free $\mathbb{Z}$-module? Justify your answer.

M4. Show that $\mathbb{Z}/3\mathbb{Z}$ is a $\mathbb{Z}/6\mathbb{Z}$-module and conclude that it is not a free $\mathbb{Z}/6\mathbb{Z}$-module.

M5. Let N be a submodule of an R-module M. Show that if N and M/N are finitely generated, then M is finitely generated.

M6. Let G be the abelian group with generators x, y, and z subject to the relations

$$\begin{aligned} 5x + 9y + 5z &= 0 \\ 2x + 4y + 2z &= 0 \\ x + y - 3z &= 0. \end{aligned}$$

Determine the elementary divisors of G and write G as a direct sum of cyclic groups.

M7. Let R be a ring and let $f : M \to N$ be a *surjective* homomorphism of R-modules, where N is a free R-module. Show that there exists an R-module homomorphism $g : N \to M$ such that $f \circ g = 1_N$. Show that $M = \mathrm{Ker}(f) \oplus \mathrm{Im}(g)$.

M8. Let R be an integral domain and let M be an R-module. A property P of M is said to be *hereditary* if, whenever M has property P, then so does every submodule N of M. Which of the following properties of M are hereditary? If a property is hereditary, give a brief reason. If it is not hereditary, give a counterexample.

(a) Free
(b) Torsion
(c) Finitely generated

M9. Let R be an integral domain. Determine if each of the following statements about R-modules is true or false. Give a proof or counterexample, as appropriate.

(a) A submodule of a free module is free.
(b) A submodule of a free module is torsion-free.
(c) A submodule of a cyclic module is cyclic.
(d) A quotient module of a cyclic module is cyclic.

M10. Let M be an R-module and let $f : M \to M$ be an R-module endomorphism which is idempotent, that is, $f \circ f = f$. Prove that $M \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$.

M11. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.

M12. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

M13. Let R be a commutative ring with 1 and let I and J be ideals of R. Prove that $R/I \cong R/J$ as R-modules if and only if $I = J$. Suppose we only ask that R/I and R/J be isomorphic as rings. Is the same conclusion valid? (*Hint:* Consider $F[X]/\langle X - a\rangle$ for $a \in F$.)

M14. Let $M \subseteq \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule of rank n. Prove that $\mathbb{Z}^n/M$ is a finite group.

M15. Let G, H, and K be finite abelian groups. If $G \times K \cong H \times K$, then prove that $G \cong H$.

M16. Let G be an abelian group and K a subgroup. For each of the following statements, decide if it is true or false. Give a proof or provide a counterexample, as appropriate.

(a) If $G/K \cong \mathbb{Z}^2$, then $G \cong K \oplus \mathbb{Z}^2$.
(b) If $G/K \cong \mathbb{Z}/2\mathbb{Z}$, then $G \cong K \oplus \mathbb{Z}/2\mathbb{Z}$.

M17. Let F be a field and let V and W be vector spaces over F. Make V and W into $F[X]$-modules via linear operators T on V and S on W by defining $X \cdot v = T(v)$ for all $v \in V$ and $X \cdot w = S(w)$ for all $w \in W$. Denote the resulting $F[X]$-modules by V_T and W_S respectively.

(a) Show that an $F[X]$-module homomorphism from V_T to W_S consists of an F-linear transformation $R : V \to W$ such that $RT = SR$.
(b) Show that $V_T \cong W_S$ as $F[X]$-modules if and only if there is an F-linear isomorphism $P : V \to W$ such that $T = P^{-1}SP$.

M18. Let $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$. Determine the elementary divisors and invariant factors of G.

M19. (a) Find a basis and the invariant factors of the submodule N of $\mathbb{Z}^2$ generated by $x = (-6, 2)$, $y = (2, -2)$ and $z = (10, 6)$.

(b) From your answer to part (a), what is the structure of $\mathbb{Z}^2/N$?

M20. Let R be a ring and let M be a free R module of finite rank. Prove or disprove each of the following statements.

(a) Every set of generators contains a basis.
(b) Every linearly independent set can be extended to a basis.

M21. Let R be a ring. An R-module N is called *simple* if it is not the zero module and if it has no submodules except N and the zero submodule.

(a) Prove that any simple module N is isomorphic to R/M, where M is a maximal ideal.
(b) Prove *Schur's Lemma:* Let $\varphi : S \to S'$ be a homomorphism of simple modules. Then either φ is zero, or it is an isomorphism.

Linear Algebra

L1. Let $\mathbb{F}$ be a field and let A be an $n \times n$ matrix with entries in $\mathbb{F}$.
  (a) State a necessary and sufficient condition on the minimal polynomial of A for A to be diagonalizable.
  (b) If $\mathbb{F} = \mathbb{C}$ and A satisfies the equation $A^3 = -A$, then show that A is diagonalizable.
  (c) If $\mathbb{F} = \mathbb{R}$, A satisfies the equation $A^3 = -A$, and A is diagonalizable, then explain what consequences this has for A.

L2. Let A be a square matrix with entries in a field $\mathbb{F}$.
  (a) If $A^n = I$ for some n, where I denotes the identity matrix, show that the eigenvalues of A are n^{th} roots of unity.
  (b) Prove that if A is nilpotent, then 0 is the only eigenvalue of A.
  (c) If 0 is the only eigenvalue of A in $\mathbb{F}$, must A be nilpotent? Justify your answer.

L3. Let V be a vector space of dimension 3 over $\mathbb{C}$. Let $\{v_1, v_2, v_3\}$ be a basis for V and let $T : V \to V$ be the linear transformation defined by $T(v_1) = 0$, $T(v_2) = -v_1$, and $T(v_3) = 5v_1 + v_2$.
  (a) Show that T is nilpotent.
  (b) Find the Jordan canonical form of T.
  (c) Find a basis of V such that the matrix of T with respect to this basis is the Jordan canonical form of T.

L4. Let V be a finite-dimensional vector space over an infinite field $\mathbb{F}$.
  (a) Prove that if $\dim V > 1$, then V contains infinitely many distinct hyperplanes (where a hyperplane is a subspace of codimension 1, the codimension of a subspace W being defined to be $\dim V - \dim W$).
  (b) Prove that V is not the union of finitely many proper subspaces.

L5. Let p be a prime number and let V be a 2-dimensional vector space over the field $\mathbb{F}_p$ with p elements.
  (a) Find the number of linear transformations $T : V \to V$.
  (b) Find the number of invertible linear transformations $T : V \to V$.

L6. Let $M = \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix}$.
  (a) Show that M is diagonalizable over the field of rational numbers.
  (b) Let $\mathbb{F}$ be a field of characteristic $p > 0$. For which values of p is M diagonalizable over $\mathbb{F}$?

L7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, with minimal polynomial $m_T(X)$ in $\mathbb{R}[X]$. Assume that $m_T(X)$ factors in $\mathbb{R}[X]$ as $f(X)g(X)$ with $f(X)$ and $g(X)$ relatively prime. Show that $\mathbb{R}^n$ can be written as a direct sum $\mathbb{R}^n = U \oplus V$, where U and V are T-invariant subspaces with $T|_U$ having minimal polynomial $f(X)$ and $T|_V$ having minimal polynomial $g(X)$.

L8. Let $\mathbb{F}$ be a field. Construct, up to similarity, all linear transformations $T : \mathbb{F}^6 \to \mathbb{F}^6$ with minimal polynomial $m_T(X) = (X-5)^2(X-6)^2$.

L9. Let V be a finite-dimensional vector space and let $S, T : V \to V$ be diagonalizable linear operators on V. The operators S and T are said to be simultaneously diagonalizable if there is a basis of V with respect to which *both* S and T are diagonal. Show that the diagonalizable linear operators S and T are simultaneously diagonalizable if and only if $ST = TS$.

L10. Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator on V.
  (a) Show that there is a basis $\mathcal{B}$ of V such that the matrix $[T]_{\mathcal{B}}$ of T with respect to $\mathcal{B}$ is upper triangular.
  (b) If S and T are two linear operators on V such that $ST = TS$, then there is a basis $\mathcal{B}$ of V such that both $[T]_{\mathcal{B}}$ and $[S]_{\mathcal{B}}$ are upper triangular.
  (c) Show that the converse of part (b) is false by showing that upper triangular matrices do not necessarily commute.

L11. Let $M_n(\mathbb{F})$ be the ring of all $n \times n$ matrices with entries in the field $\mathbb{F}$, and let $\det : M_n(\mathbb{F}) \to \mathbb{F}$ be the determinant function.
  (a) Show that det is surjective.
  (b) Now assume that $\mathbb{F}$ is a *finite* field. Prove that all *nonzero* elements of $\mathbb{F}$ are taken on the same number of times by the determinant function.

L12. Recall that a complex matrix $A = [a_{ij}]$ is *Hermitian* if $A^* = A$, where A^* denotes the conjugate transpose $\overline{A}^t$ of A. That is, A is Hermitian if and only if $a_{ij} = \overline{a}_{ji}$ for all i, j. You may assume the fact that the Hermitian $n \times n$ matrices, under the usual addition and scalar multiplication of matrices, form a real vector space H_n.

(a) Find a basis of H_3 and determine its dimension as a real vector space. What is the dimension of H_n as a real vector space?

(b) Is H_n a vector space over $\mathbb{C}$? Explain.

L13. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent linear transformation.

(a) How is $\dim \operatorname{Ker} T$ related to the Jordan normal form of T? How is the minimal polynomial related to the Jordan normal form?

(b) Let T, $S : \mathbb{C}^6 \to \mathbb{C}^6$ be nilpotent linear transformations such that S and T have the same minimal polynomial and $\dim \operatorname{Ker} T = \dim \operatorname{Ker} S$. Show that S and T have the same Jordan form.

(c) Show that there are nilpotent linear transformations T, $S : \mathbb{C}^8 \to \mathbb{C}^8$ such that S and T have the same minimal polynomial and $\dim \operatorname{Ker} T = \dim \operatorname{Ker} S$, but S and T have different Jordan forms. That is, part (b) is false if 6 is replaced by 8.

L14. (a) Define *basis* of a finite dimensional vector space.

(b) Prove that any two bases of a finite-dimensional vector space have the same number of elements.

L15. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces V and W. Show that $\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$.

L16. Let A be an $m \times n$ matrix with entries in a field $\mathbb{F}$. Prove that the column rank of A and the row rank of A are equal. Recall that the column rank of A is the maximal number of linearly independent columns. The row rank is defined similarly.

L17. Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

L18. Let N be a linear operator on an n-dimensional vector space V with $n > 1$. Assume that $N^n = 0$ but $N^{n-1} \neq 0$. Prove that there is no linear operator T on V with $T^2 = N$.

L19. Let $T : V \to W$ be a linear transformation between finite-dimensionanl vector spaces V and W. Prove that there is a basis $\mathcal{B}$ of V, and a basis $\mathcal{C}$ of W so that the matrix $[T]_{\mathcal{B},\mathcal{C}}$ of T with respect to $\mathcal{B}$ and $\mathcal{C}$ has the block form

$$[T]_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

L20. (a) Let $A \in \mathrm{GL}(n, \mathbb{C})$ generate a cyclic subgroup of finite order. Show that A is diagonalizable.

(b) By considering $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$ show that the result is false if $\mathbb{C}$ is replaced by $\mathbb{R}$.

L21. Let $\mathbb{F}$ be a field and let

$$0 \longrightarrow V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_n} V_{n+1} \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces and linear transformations over $\mathbb{F}$. This means that T_1 is injective, T_n is surjective, and $\operatorname{Im}(T_i) = \operatorname{Ker}(T_{i+1})$ for $1 \leq i \leq n-1$. Show that

$$\sum_{i=1}^{n-1} (-1)^{i+1} \dim V_i = 0.$$

L22. An algebraic integer is a complex number which is a root of a monic polynomial with integer coefficients. Show that every algebraic integer is an eigenvalue of a matrix $A \in M_n(\mathbb{Z})$ for some n.

L23. Let V be a vector space over a field $\mathbb{F}$ and let $\mathcal{S}$ be a linearly independent subset of V. Prove that there exists a basis $\mathcal{B}$ of V containing $\mathcal{S}$. (*Hint:* Apply Zorn's Lemma.)

L24. Let S and T be linear transformations between finite-dimensional vector spaces V and W over the field $\mathbb{F}$. Show that $\operatorname{Ker} S = \operatorname{Ker} T$ if and only if there is an invertible operator U on W such that $S = UT$.

L25. (a) Let $A \in M_n(\mathbb{C})$. If the eigenvalues of A are distinct, show that A is diagonalizable, i.e., there is an invertible matrix P such that $P^{-1}AP$ is diagonal.
(b) What is the condition on $A \in M_n(\mathbb{C})$ so that A is unitarily diagonalizable? Simply state the condition and quote (do not prove) the theorem which justifies your answer. Recall that A is unitarily diagonalizable if there is a unitary matrix U such that U^*AU is diagonal.
(c) Give an example of a complex matrix A which is diagonalizable, but not unitarily diagonalizable.

L26. Find a square root of the matrix

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}.$$

L27. Suppose that the vector space V is the sum of subspaces W_1 and W_2. That is $V = W_1 + W_2$, but we are *not* assuming that the sum is direct. Let $\mathcal{S}$ be the vector space of all linear operators $T : V \to V$ such that $T(W_1) \subseteq W_1$ and $T(W_2) \subseteq W_2$. Compute the dimension of $\mathcal{S}$ as a function of $n = \dim V$, $m_1 = \dim W_1$, and $m_2 = \dim W_2$.

L28. Find the Jordan form of all real matrices with characteristic polynomial $(X-1)^5(X+1)$ and minimal polynomial $(X-1)^2(X+1)$.

L29. In this problem we will work with complex $n \times n$ matrices.
(a) Define *unitary* matrix, *Hermitian* matrix, and *normal* matrix.
(b) State the *Spectral Theorem* for normal matrices.
(c) Show that a normal matrix is Hermitian if and only if all the eigenvalues are real.

L30. Suppose that A is a 3×3 complex matrix such that

$$A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Show that A is diagonalizable.

L31. Let T be a linear operator on a finite-dimensional complex inner product space V. Show that T is a normal operator if and only if $T^* = g(T)$ where $g(X)$ is a complex polynomial. (*Hint:* Use the spectral theorem and Lagrange interpolation.)

L32. Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$. Then A is a symmetric matrix with rational entries. Find a rational invertible matrix P so that P^tAP is diagonal.

L33. Determine the signature of the real quadratic form with matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 9 & 5 \\ 4 & 5 & 0 \end{bmatrix}.$$

L34. Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Show that A is positive definite if and only if A is invertible and A^{-1} is positive definite.

L35. If A is a real symmetric matrix prove that there is an $\alpha \in \mathbb{R}$ such that $A + \alpha I_n$ is positive definite.

L36. Prove that the eigenvalues of a real symmetric matrix are real numbers.

L37. Let V be a finite-dimensional real vector space and let $T : V \to V$ be a nilpotent transformation (i.e. $T^j = 0$ for some positive integer j).
(a) Find the eigenvalues of T.
(b) Is $I - T$ invertible, where $I : V \to V$ is the identity transformation? Explain fully.
(c) Give an example of two non-similar linear transformations A and B on the same finite dimensional vector space V, having identical characteristic polynomials and identical minimal polynomials.

L38. Let $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the linear transformation expressed relative to the standard basis $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ by the matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

(a) Find a basis for $\mathrm{Ker}(T)$.
(b) Find a basis for $\mathrm{Im}(T)$.
(c) Find the matrix for T expressed in the basis $\mathbf{f}_1 = (-1, 1, 0)$, $\mathbf{f}_2 = (0, 1, -1)$, $\mathbf{f}_3 = (1, 0, 1)$.

L39. Find the characteristic polynomial, minimal polynomial, and Jordan canonical form of the linear transformation T with matrix
$$\begin{bmatrix} 4 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & 0 \end{bmatrix}.$$

L40. If A is a diagonalizable complex $n \times n$ matrix, show that A has a square root. That is, there is a complex $n \times n$ matrix B such that $B^2 = A$.

L41. Prove that the matrix equation $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not solvable for $A \in M_2(\mathbb{C})$ (the ring of 2×2 complex matrices).

L42. (a) Let $A \in M_n(\mathbb{C})$ satisfy the equation $A^k = I_n$ for some $k \geq 1$, where I_n is the $n \times n$ identity matrix. Show that A is diagonalizable.
(b) Show that the result of part (a) is not true if the field $\mathbb{C}$ is replaced by a field $\mathbb{F}$ of characterisitic $p > 0$. (*Hint:* Check that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{F})$ satisfies $A^p = I$.)

L43. Let V be the vector space of polynomials $p(X) \in \mathbb{C}[X]$ of degree ≤ 4. Define a linear transformation $T : V \to V$ by $T(p(X)) = p''(X)$ (the second derivative of the polynomial $p(X)$). Compute the characteristic polynomial, minimal polynomial, and Jordan canonical form of the linear transformation T.

L44. Let V and W be finite-dimensional vector spaces over a field $\mathbb{F}$, and let $\mathrm{Hom}(V, W)$ denote the vector space of linear transformations from V to W. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V and let $\mathcal{C} = \{w_1, \ldots, w_m\}$ be a basis of W.
(a) Write down a basis of $\mathrm{Hom}(V, W)$ and verify that this is a basis.
(b) What is the dimension of $\mathrm{Hom}(V, W)$ over $\mathbb{F}$?

L45. Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Prove that the dual space of V is isomorphic to V.

L46. Let A and B be $n \times n$ matrices with entries in a field $\mathbb{F}$.
(a) Show that AB and BA have the same characteristic values (*not* counting multiplicities).
(b) If A and B are similar, show that A and B have the same characteristic polynomials.

L47. Let p be a prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements, $V = \mathbb{F}_p^4$ (a 4-dimensional vector space over $\mathbb{F}_p$), and W the subspace of V spanned by the three vectors $\mathbf{a}_1 = (1, 2, 2, 1)$, $\mathbf{a}_2 = (0, 2, 0, 1)$, and $\mathbf{a}_3 = (-2, 0, -4, 3)$. Find $\dim_{\mathbb{F}_p} W$. (Note that this dimension depends on p.)

L48. Let V be the real inner product space $\mathbb{R}^n$ with the standard inner product $\langle\,,\rangle$. Show that a linear operator $T : V \to V$ preserves the length of each vector if and only if T is an *isometry*, that is, $\langle T(v), T(w)\rangle = \langle v, w\rangle$, for all $v, w \in V$.

L49. Prove that if the columns of a real $n \times n$ matrix form an orthonormal basis of $\mathbb{R}^n$, then the rows do too.

L50. Let A and B be real $n \times n$ matrices. Prove that $A = B$ if and only if $X^tAY = X^tBY$ for all $n \times 1$ real matrices X, Y.

L51. Let V denote the vector space of real $n \times n$ matrices. Prove that $\langle A, B\rangle = \mathrm{Trace}(A^tB)$ is a positive definite bilinear form on V. Find an orthonormal basis for this form.

L52. (a) Prove that every real square matrix is the sum of a symmetric matrix and a skew-symmetric matrix ($A^t = -A$) in exactly one way.
(b) Let $\langle\,,\rangle$ be a bilinear form on a finite-dimensional real vector space V. Show that there is a symmetric form $(\,,)$ and a skew-symmetric form $[\,,]$ so that $\langle\,,\rangle = (\,,) + [\,,]$.

L53. Prove that the only real $n \times n$ matrix that is orthogonal, symmetric, and positive definite is the identity.

L54. Let A be an $n \times n$ real symmetric matrix, and let T be the linear operator on $\mathbb{R}^n$ whose matrix in the standard basis is A.
(a) Prove that $\mathrm{Ker}(T)$ is orthogonal to $\mathrm{Im}(T)$ and that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$.
(b) Prove that T is an orthogonal projection onto $\mathrm{Im}(T)$ if and only if, in addition to being symmetric, $A^2 = A$.

L55. Let A be a normal complex $n \times n$ matrix.

(a) Show that $\operatorname{Ker} A = \operatorname{Ker}(\overline{A}^t)$.

(b) Show that $\operatorname{Ker} A = (\operatorname{Im} A)^\perp$ in the standard complex inner product on $\mathbb{C}^n$.