MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I - ANALYSIS

August 2000

Directions: The test is divided into five sets of problems (A), (B), (C), (D), and (E). **Do problem (A)** and select **exactly one problem** from *each of the four sets (B) - (E).* Please answer each problem on a seperate sheet of paper. Turn in only the five problems you wish to have graded.

Problem Set (A). Please answer each problem with 'true' or 'false' only. Do not explain.

(a) Every Lipschitz continuous function $f : [0,1] \to \mathbb{R}$ is absolutely continuous.

(b) Every Lipschitz continuous function $f : [0,1] \to L^1[0,1]$ is differentiable almost everywhere.

(c) $M := \{f \in C[0,1] : f(1) > 0\}$ is open in the space $C[0,1]$ of real continuous functions on $[0,1]$ equipped with the sup norm.

(d) $L^1[0,1] \subset L^2[0,1]$.

(e) A function $f : [0,1] \to \mathbb{R}$ is continuous if it is the uniform limit of step functions.

(f) The product of three uniformly continuous functions in $C[0,1]$ may not be uniformly continuous.

(g) A continuous function $f : \mathbb{R} \to \mathbb{R}$ can be approximated uniformly on $\mathbb{R}$ by polynomials.

(h) The Maclaurin series of every infinitely often differentiable function $f : [-1,1] \to \mathbb{R}$ converges pointwise to f in some neighborhood of 0.

(i) Let $F : [0,1] \to \mathbb{R}$ be of bounded variation. Then F is differentiable almost everywhere, F' is integrable, and $F(1) - F(0) = \int_0^1 F'(s)\,ds$.

(j) Every bounded sequence in $C[0,1]$ has a uniformly convergent subsequence.

Problem Set (B).

B.1 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

B.2 Let k be a measurable function on $\mathbb{R}^2$ such that $\int_{\mathbb{R}}(\int_{\mathbb{R}} |k(x,y)|^q dy)^{p/q} dx < \infty$ for some $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} + 1$. Show that

$$(Tf)(x) := \int_{\mathbb{R}} k(x,y)f(y)dy$$

defines a continuous linear map $T : L_p(\mathbb{R}) \to L_p(\mathbb{R})$.

Problem Set (C).

C.1 Let $f_n(x) = \dfrac{n}{x(\ln x)^n}$ for $x \geq e$ and $n \in \mathbb{N}$.

(a) For which $n \in \mathbb{N}$ does the Lebesgue integral $\int\limits_e^\infty f_n(x)\,dx$ exist?

(b) Determine $\lim_{n\to\infty} f_n(x)$ for $x > e$.

(c) Does the sequence $(f_n)_{n\in\mathbb{N}}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

C.2 Find and justify the limits.

(a) $\displaystyle\lim_{n\to\infty}\int_0^n \frac{\sin x}{1+nx^2}\,dx.$

(b) $\displaystyle\lim_{n\to\infty}\int_0^{e^n} \frac{x}{1+nx^2}\,dx.$

(c) $\displaystyle\lim_{n\to\infty}\int_0^{\infty} \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)\,dx.$

Problem Set (D).

D.1 Let $f_N(x) = \sum_{n=1}^N a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^\infty n a_n$ converges absolutely, show that $(f_N)_{N\in\mathbb{N}}$ converges uniformly to a function f on $\mathbb{R}$, and that $(f_N')_{N\in\mathbb{N}}$ converges uniformly to f' on $\mathbb{R}$.

D.2 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that given x and h then $f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi)$ for some ξ.

(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

Problem Set (E).

E.1 Let (X, d) be a complete metric space. A mapping $F : X \to X$ is said to be a contraction if there is a constant $r < 1$ such that $d(F(u), F(v)) \le r \cdot d(u, v)$ for all $u, v \in X$. Recall that the contraction mapping principle states that every contraction has a unique fixed point in X.

(a) Let $g \in C[0,1]$ with $\int_0^1 |g(s)|\,ds \le r < 1$. Use the contraction mapping principle to show that, for all $f \in C[0,1]$, there exists a unique solution $u = u(\cdot) \in C[0,1]$ of the equation

$$u(t) = \int_0^t g(t-s)u(s)ds + f(t), \qquad 0 \le t \le 1. \tag{$*$}$$

(b) Show that the operator A which assigns to each $f \in C[0,1]$ the unique solution u of the equation $(*)$ is a linear operator from $C[0,1]$ into $C[0,1]$.

(c) Use the 'Closed Graph Theorem' to show that A is a continuous linear operator and use the continuity of A to show that the solutions u of $(*)$ depend continuously on the forcing terms f.

E.2 Let $f_n(x) := \dfrac{x^n}{n!}e^{-x}$ for $n \in \mathbb{N}_0$.

(a) Show that $\lim_{x\to\infty} f_n(x) = 0$ for all $x > 0$.

(b) Show that $f_n \in L^1(0,\infty)$ with $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}_0$.

(c) Show that $\lim_{k\to\infty}\int_0^k \frac{x^n}{n!}(1-\frac{x}{k})^k\,dx = 1$ for all $n \in \mathbb{N}_0$.