MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I - ANALYSIS

January 2001

Directions: The test is divided into five sets of problems (A), (B), (C), (D), and (E). Do problem (A) and select one problem from each of the four sets (B) - (E). Please answer each problem on a separate sheet of paper. Turn in only the five problems you wish to have graded.

Problem Set (A). Please answer each problem with 'true' or 'false' only. Do not explain.

(a) Let f_n be a sequence of continuous functions on $\mathbb{R}$ such that f_n converges uniformly to f on $\mathbb{R}$ and $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\,dt = 2$. Then $\int_{-\infty}^{\infty} f(t)\,dt = 2$.

(b) Let f_n be Lebesgue integrable functions on $[0,1]$. If $\lim_{n\to\infty} \int_0^1 f_n(t)\,dt = 0$, then f_n converges pointwise to zero almost everywhere.

(c) Let f be a continuous function on $\mathbb{R}$ such that the Lebesgue integral $\int_{-\infty}^{\infty} f(t)\,dt$ exists. Then $\lim_{t\to\infty} f(t) = 0$.

(d) $L^\infty[0,1] \subset L^2[0,1] \subset L^1[0,1]$.

(e) The 'Closed Graph Theorem' implies that every linear operator with domain and range in a Banach space X is a bounded linear operator from X to X if its graph is closed in the product space $X \times X$.

(f) Let $f(0) := 0$ and $f(t) := \frac{1}{\sqrt{t}}$ for $t > 0$. Then f is Riemann integrable on $[0,1]$, and $\int_0^1 f(t)\,dt = 2\sqrt{t}|_0^1 = 2$.

(g) The set of all continuous functions f with $\int_0^1 f(t)\,dt = 0$ is compact in the space $C[0,1]$ of all real continuous functions on $[0,1]$ equipped with the sup norm.

(h) The linear span of the functions $r_n(t) := \frac{1}{1+nt}$ $(0 \le n \in \mathbb{N})$ is dense in $C[0,1]$ equipped with the sup norm.

(i) Every continuous function on $\mathbb{R}$ can be approximated uniformly on $\mathbb{R}$ by polynomials.

(j) The polynomials are dense in $L^1[0,1]$.

Problem Set (B).

B.1 Let X be the space of continuous real functions on $[0,1]$ with norm

$$\|f\|_0 := \sup_{t\in[0,1]} \left|\int_0^t f(s)\,ds\right|.$$

(a) Show that the functions $f_n(t) := \sin(nt)$ converge to zero in X as $n \to \infty$.
(b) Show that X is not a Banach space.

B.2 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0. \end{cases}$$

(a) Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1, 2, 3, \cdots$. Show details.
(b) Is f equal to the sum of a power series expansion in an open neighborhood of 0?
(c) Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0,1)$.

Problem Set (C).

C.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives. By considering the Taylor expansion of f, show that:

$$\|f'\|_\infty \le \frac{1}{h}\|f\|_\infty + h\|f''\|_\infty$$

for every $h > 0$. By minimizing over h, show that $\|f'\|_\infty \le 2\sqrt{\|f\|_\infty \|f''\|_\infty}$, where $\|g\|_\infty$ denotes $\sup_{x\in\mathbb{R}} |g(x)|$.

C.2 Let $f(x) = x^2 \sin(\frac{1}{x}), g(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(x) = g(x) = 0$ for $x = 0$. Show:

(a) f and g are differentiable everywhere (including $x = 0$),
(b) f is bounded variation on the interval $[-1,1]$, but g is not.
(c) Let $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Is f of bounded variation on $[-1,1]$?

Problem Set (D).

D.1 Let f be a positive function on $(0,1]$ such that f is Riemann integrable on $[\epsilon,1]$ for all $\epsilon \in (0,1)$, but $\lim_{t\to 0^+} f(t) = \infty$. Assume that the improper Riemann integral $(R)\int_0^1 f(t)\,dt$ exists. Show that f is Lebesgue integrable, and that the Lebesgue integral $\int_0^1 f(t)\,dt$ coincides with the improper Riemann integral $(R)\int_0^1 f(t)\,dt$.

D.2 Let f be a differentiable function on $[-1,1]$. Prove that $\lim_{\epsilon\to 0}\int_{\epsilon<|t|\le 1}\frac{1}{t}f(t)\,dt$ exists. Does the Lebesgue integral $\int_{-1}^{1}\frac{1}{t}f(t)\,dt$ exist? Explain.

Problem Set (E).

E.1 Let $g_n = n\chi_{[0,n^{-3}]}$. Show that $\int_0^1 f(t)g_n(t)\,dt \to 0$ as $n\to\infty$ for all $f \in L^2[0,1]$, but not for all $f \in L^1[0,1]$.

E.2 Prove that $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)\sin(nt)\,dt = 0$ for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Do the problem first for step functions.)