Measure and Integration

Instructions: *Work out all the problems* 1–4. *Choose one problem from* 5–7 *and also one problem from* 8–10*, making a total of six (6) problems. You have three and one-half hours to complete this test.*
Note: In all of the problems λ *denotes Lebesgue measure on* $\mathbb{R}$.

1. Let $(X, \mathbb{X}, \mu)$ be a measure space and let $f \in L^1(X, \mu)$, $f \geq 0$.
a. Show that the set $\{x \in X \mid f(x) = \infty\}$ has measure zero.
b. Show that the set $\{x \in X \mid f(x) > 0\}$ is σ-finite.

2. a. Let $(X, \mathbb{X}, \mu)$ be a measure space. Let (f_n) be a sequence in $L^1(X, \mu)$ and suppose that $f_n \to f$ uniformly on X.
Show that if $\mu(X) < \infty$ then $f \in L^1(X, \mu)$ and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

b. Give an example where $\mu(X) = \infty$ and a sequence (f_n) in $L^1(X, \mu)$ converging uniformly to a function $f \in L^1(X, \mu)$, but such that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \neq \int_X f \, d\mu.$$

3. Denote by $d\lambda$ Lebesgue measure on $\mathbb{R}$. Evaluate the following limits. Explain your answers.

a. $\lim_{n\to\infty} \int_0^\infty (1 + x/n)^n \, e^{-2x} \cos(x/n) \, d\lambda(x)$.
b. $\lim_{n\to\infty} \int_0^1 (1 + nx^2) \, (1 + x^2)^{-n} \, d\lambda(x)$.

4. Let $(X, \mathbb{X}, \mu)$ be a σ-finite measure space. Let $1 < p \leq \infty$. Show that $L^p(X, \mu) \subset L^1(X, \mu)$ if and only if $\mu(X) < \infty$.

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5. Let f be a compactly supported continuous function on $\mathbb{R}$. Then

$$\lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{nx^2}{2}} f(x) \, d\lambda = f(0) \, .$$

6. Let $f_n = \chi_{[n,n+1)}$, $n \in \mathbb{N}$. Show that $f_n \to 0$ pointwise but that f_n does not converges to zero in measure (with respect to λ).

7. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable monotone function. Let λ_F be the Legesgue-Stieltjes measure on $\mathbb{R}$ determined by

$$\lambda_F([a,b)) = F(b) - F(a)\,.$$

Show that if $f \in L^1(\mathbb{R}, \lambda_F)$ then $fF' \in L^1(\mathbb{R}, \lambda)$ and

$$\int_{\mathbb{R}} f \, d\lambda_F = \int f F' \, d\lambda\,.$$

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8. Let $(X, \mathbb{X}, \mu)$ be a measure space and let $\mathbb{Y} \subset \mathbb{X}$ be a σ-algebra. Let ν be the restriction of μ to $\mathbb{Y}$ and suppose the measure ν is σ-finite. Show that if $f \in L^1(X, \mu)$ then there exists a function $g \in L^1(X, \nu)$ such that

$$\int_E f(x)\, d\mu(x) = \int_E g(x)\, d\nu(x)$$

for all $E \in \mathbb{Y}$.

9. Let $(X, \mathbb{X}, \mu)$ be a measure space. Let $1 \le p_1 < p_2 < \infty$. Suppose that $f \in L^{p_1}(X, \mu) \cap L^{p_2}(X, \mu)$. Show that $f \in L^q(X, \mu)$ for all $q \in [p_1, p_2]$.

10. Let

$$f(x,y) := \begin{cases} \frac{x^2 - y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is not integrable on $[0,1] \times [0,1]$ with respect to $\lambda \times \lambda$.