Analysis Test Bank

Preliminary Version, September 1999

1. Metric and normed spaces, completeness and completion
2. Continuous linear transformations and functions, closed graph theorem, open mapping theorem
3. Continuous functions: uniform convergence, principle of uniform boundedness, Stone-Weierstrass, compactness
4. Differentiable functions: Jacobians, inverse and implicit functions, power series
5. Functions of bounded variation
6. Riemann integral, Lebesgue integral (via completion), convergence theorems
7. Absolutely continuous functions and the fundamental theorem of calculus
8. Basic properties of L^p-spaces, Riesz representation for L^p-spaces

Suggested Reading

D.S. Bridges: Foundations of Real and Abstract Analysis. Springer, 1997

A. Browder: Mathematical Analysis. Springer, 1996.

I.P. Natanson: Theory of Functions of a Real Variable, Vol. 1, 1955.

H.L. Royden: Real Analysis, Macmillan, 1988.

W. Rudin: Principles of Mathematical Analysis. McGraw-Hill, 1953.

G.F. Simmons: Introduction to Topology and Modern Analysis. McGraw-Hill, 1963.

R.L. Wheeden and A. Zygmund: Measure and Integral: An Introduction to Real Analysis. Marcel Dekker, 1977.

1. Metric and normed spaces, completeness and completion

1.1 Let X be a compact set and $C(X)$ the space of real continuous functions on X equipped with the sup norm. Let $M = \{f \in C(X)$ such that $f(x) > 0$ for all $x \in X\}$. Show that M is an open subset of $C(X)$.

1.2 Let X be the normed linear space obtained by putting the norm $\|f\|_0 := \sup_{t\in[0,1]} |\int_0^t f(s)\,ds|$ on the set of continuous real functions on $[0,1]$.

(a) Show that the functions $f_n(t) := \sin(nt)$ converge to zero in X.

(b) Show that X is not a Banach space.

(c) Let $f : [0,1] \to \mathbb{R}$ be continuous and extend f to $[0,\infty)$ by setting $f(t) := f(1)$ for $t > 1$. Show that the differential quotients $D_h : t \to \frac{f(t+h)-f(t)}{h}$ converge uniformly on [0,1] as $h \to 0^+$ if f is continuously differentiable. Show that the differential quotients D_h are Cauchy in X as $h \to 0^+$ for any continuous $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$; i.e., for all $\epsilon > 0$ there exists $h_0 > 0$ such that $\|D_h - D_k\|_0 < \epsilon$ for all $0 < h, k < h_0$.

1.3 For $0 < \alpha \leq 1$ consider the space L_α of all functions on $[0,1] \to \mathbb{R}$ such that

$$\|f\|_\alpha := |f(0)| + \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty.$$

(a) Show that L_α is a Banach space.

(b) Show that each element in L_α is absolutely continuous.

2. Continuous linear transformations and functions closed graph theorem, open mapping theorem

2.1 (a) Let X be a complete metric space. A mapping $F : X \to X$ is said to be a contraction if there is a constant $r < 1$ such that $d(F(u), F(v)) \le r \cdot d(u, v)$ for all $u, v \in X$. Given a contraction F and a point $u_0 \in X$, define a sequence $(u_k)_{k \in N}$ in X by $u_{k+1} = F(u_k)$. Show that $d(u_{k+1}, u_k) \le r^k d(u_1, u_0)$ and prove that the sequence (u_k) converges to the unique fixed point of F.

(b) Let $g \in C[0, 1]$ with $\int_0^1 |g(s)|\, ds \le r < 1$. Use part (a) to show that, for all $f \in C[0, 1]$, there exists a unique solution $u = u(\cdot) \in C[0, 1]$ of the equation

$$u(t) = \int_0^t g(t - s)u(s)ds + f(t),\ \ 0 \le t \le 1. \qquad (*)$$

(c) Show that the operator A which assigns to each $f \in C[0, 1]$ the unique solution u of the equation $(*)$ is a linear operator from $C[0, 1]$ into $C[0, 1]$.

(d) Use the 'Closed Graph Theorem' to show that A is a continuous linear operator.

(e) Show that the solutions u of $(*)$ depend continuously on the forcing terms f.

2.2 Let k be a measurable function on $I\!R^2$ such that $\int_{\mathbf{R}}(\int_{\mathbf{R}} |k(x, y)|^q dy)^{p/q} dx < \infty$ for some $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} + 1$. Show that

$$(Tf)(x) := \int_{\mathbf{R}} k(x, y)f(y)dy$$

defines a continuous linear map $T : L_p(I\!R) \to L_p(I\!R)$.

2.3 Let X be the normed linear space obtained by putting the norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$ on the set of continuous real functions on $[0, 1]$.

(a) Show that X is not a Banach space. [$\to$ **1**]

(b) Show that the linear functional $\Lambda f = f(1/2)$ is not bounded.

3. Continuous functions: uniform convergence, principle of uniform boundedness, Stone-Weierstrass, compactness

3.1 A subset S of $\mathbb{R}$ is of type F_σ if S is the countable union of closed sets.

(a) Let f be any function from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set of points of discontinuity of f is of type F_σ.

(b) Can a function from $\mathbb{R}$ to $\mathbb{R}$ be continuous on the rationals and discontinuous on the irrationals? What if the roles of the rationals and irrationals are interchanged?

(c) Briefly explain why there are continuous, nowhere differentiable functions on $\mathbb{R}$.

3.2 (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$ $(n \in \mathbb{N})$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on $\mathbb{R}$ but not uniformly convergent on $\mathbb{R}$.

(b) Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{1}{1+nx^2}$ $(n \in \mathbb{N})$. Show that $(f_n)_{n\in\mathbb{N}}$ is a bounded subset of $C[0,1]$ and that no subsequence of $(f_n)_{n\in\mathbb{N}}$ converges in $C[0,1]$.

3.3 Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{1+nx^2}$ $(n \in \mathbb{N})$.

(a) Show that $\sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2\sqrt{n}}$. Conclude that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$ to a function f. What is f?

(b) Show that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is true if $x \neq 0$, but false if $x = 0$.

3.4 Identify all subsets of $[0,1]$ on which $\sum_{n=0}^{\infty} x^n$ converges uniformly. Explain.

3.5 (a) Show that every continuous function on $[0,1]$ is a uniform limit of step functions.

(b) Is the converse true? (A step function is finite linear combination of characteristic functions of intervals.)

3.6 Show that every continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ can be uniformly approximated by polynomials $p(s,t) = \alpha_0 + \sum_{n,m=1}^{N} \alpha_{nm} t^n x^m$, where $\alpha_{nm} \in \mathbb{R}$ and $N \in \mathbb{N}$. Is the same result true for continuous functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$?

3.7 Prove or disprove: The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.

3.8 Let $\chi_{[-n,n]}(\cdot)$ denote the characteristic function of the interval $[-n,n]$ $(n \in \mathbb{N})$. Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x) \sin(\frac{\pi x}{n})$ $(x \in \mathbb{R})$.

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

(b) Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

3.9 Prove or disprove the following two statements:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise everywhere on $\mathbb{R}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges to a continuous function on $\mathbb{R}$.

3.10 Let $f(x) = \begin{cases} x^2, \text{ if } x \text{ is irrational} \\ 0, \text{ if x is rational} \end{cases}$ Show that f is continuous at only one point, and that f is differentiable there.

3.11 Let $C_{0,0}[0,1]$ be the space of all continuous real functions f on the internal $[0,1]$ satisfying $f(0) = f(1) = 0$. Let $P_{0,0}$ be the subspace of polynomials in $C_{0,0}[0,1]$. Show that $P_{0,0}$ is dense in $C_{0,0}[0,1]$ in the sup norm.

3.12 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) := (x/n)e^{-(x/n)}$ $(n \in \mathbb{N})$.

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_{n\in N}$ converges uniformly to f on $[0,a]$ for any non-negative real number a. Does the sequence converge uniformly to f on $[0,\infty)$? Justify your answer.

(b) Show that $f(x) = \lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a f(x)\,dx$, but that $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx$.

3.13 Let $I = [0,1]$. Suppose f is a continuous real-valued function on $I \times I$. Show that f can be uniformly approximated by functions of the from $\sum_{i=1}^n f_i(x)g_i(y)$ where f_i and g_i are continuous real-valued function on I.

3.14 Show: if $f \in C[0,1]$ and $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

3.15 Let $(x_n)_{n\in N}$ be a sequence of real numbers. Prove that the following are equivalent.

(i) $\lim_{n\to\infty} x_n = a$.

(ii) Every subsequence of $(x_n)_{n\in N}$ contains a subsequence that converges to a.

3.16 Show that the function $f(x) = 2^{-x} + 3\cdot 2^{-3x} + \ldots + (2n+1)2^{-(2n+1)x} + \ldots$ is continuous on $(0,\infty)$.

3.17 (a) Prove or disprove: A continuous function on the interval $[0,\infty)$ can be approximated uniformly by polynomials.

(b) Prove or disprove: If f and g are both functions from $\mathbb{R}$ into $\mathbb{R}$, and $\lim_{t\to a} g(t) = b$ and $\lim_{t\to b} f(t) = c$, then $\lim_{t\to a} f(g(t)) = c$.

3.18 Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable. Prove that for any $\epsilon > 0$, there exists a polynomial $P(x)$ such that $\|f - P\|_\infty < \epsilon$ and $\|f' - P'\|_\infty < \epsilon$. Here $\|\cdot\|_\infty$ denotes the sup-norm.

3.19 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

3.20 Let $f_n : [1,\infty) \to \mathbb{R}$ be defined by $f_n(x) := \frac{n+1}{n}e^{-nx}$ $(n \in \mathbb{N})$. Show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuous function.

3.21 $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := (\sin x)^n$ $(n \in \mathbb{N})$. Does $(f_n)_{n\in \mathbb{N}}$ converge uniformly?

4. Differentiable functions: Jacobians, inverse and implicit functions, power series

4.1 Does $e^z(x^2+y^2+z^2) - \sqrt{1+z^2} + y = 0$ have a solution $z = f(x,y)$ which is continuous at $x = 1, y = 0$ and $f(1,0) = 0$? Explain carefully!

4.2 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that given x and h then $f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi)$ for some ξ.

(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

4.3 Let $f(x,y) = xy - \cos y + x^2 + 1$. At what points on the set $\{(x,y) : f(x,y) = 0\}$ does the condition $f(x,y) = 0$ fail to define either x as a function of y or y as a function of x?

4.4 Prove or give a counterexample: If f is a uniform limit of polynomials on $[-1,1]$, then the Maclaurin series of f converges to f in some neighborhood of 0.

4.5 Can the surface whose equation is $xy - z\log y + e^{xz} = 1$ be represented in the form $z = f(x,y)$ in a neighborhood of $(0,1,1)$? In the form $y = g(x,z)$ in a neighborhood of $(0,1,1)$?

4.6 Let $f(x) = z^k \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) If $k = 2$, show that f is differentiable everywhere but f' fails to be continuous at some point.

b If $k = 3$, does f have a second derivative at all points? If so, is f'' a continuous function? Give your reasons.

4.7 Let f be defined on $\mathbb{R}^3$ by $f(x,y,z) = x^2+4y^2-2yz-z^2$. Show that $f(2,1,-4) = 0$ and $f_z(2,1,-4) \neq 0$, and that there exists therefore a differentiable function g in a neighborhood of $(2,1)$ in $\mathbb{R}^2$, such that $f(x,y,g(x,y)) = 0$. Find $g_x(2,1)$ and $g_y(2,1)$. What is the value of $g(2,1)$?

4.8 Suppose that a function f is defined on $(0,1]$ and has a finite derivative with $|f'(x)| < 1$. Define $a_n := f(1/n)$ for $n = 1,2,3,\ldots$. Show that $\lim_{n\to\infty} a_n$ exists.

4.9 Prove or disprove: The series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+n}{n^2}$ is uniformly convergent on $[-1,1]$.

4.10 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$$

(a) Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1,2,3,\cdots$. Show details.

(b) Does f have a power series expansion at 0?

(c) Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0,1)$.

4.11 Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$ if the partial derivatives $f_{x_1},\ldots,f_{x_n}$ exist and are bounded in a neighborhood of x.

4.12 Let $f_N(x) = \sum_{n=1}^{N} a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^{\infty} na_n$ converges absolutely, show that $(f_N)_{N\in\mathbf{N}}$ converges uniformly to a function f on $\mathbb{R}$, and that $(f'_N)_{N\in\mathbf{N}}$ converges uniformly to f' on $\mathbb{R}$.

4.13 Let f be a twice continuously differentiable real-valued function on $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a critical point of f if all partial derivatives of f vanish at x (i.e., $\bigtriangledown f(x) = 0$), a critical point x is nondegenerate if the $n \times n$ matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$$

is nonsingular. Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points. (i.e., the nondegenerate critical points are isolated.)

4.14 Show that the power series in x for the function $f(x) = e^{ax}\cos(bx)$ $(a, b \in \mathbb{R})$ has either no zero coefficients or infinitely many zero coefficients.

4.15 Let f be continuous function on $[0, 1]$. Define

$$I_n = \prod_{j=1}^{n}\left[1 + \frac{1}{n} f\left(\frac{j}{n}\right)\right]$$

for every integer $n \geq 1$. Determine the limit $\lim_{n\to\infty} I_n$.

4.16 Show that a function $f(x) = e^{-x} + 2e^{-2x} + \ldots + ne^{-nx} + \ldots$ is continuous on $(0, \infty)$.

4.17 Let $f(x) = e^{-1/x^2}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. Verify, using induction, that for each positive integer k, there is a polynomial p_k such that $f^{(k)}(x) = f(x)p_k(x)x^{-3k}$ for all $x > 0$. Show that the function $\phi : x \to f(1 + x)f(1 - x)$ has the following properties:

(i) ϕ is smooth (i.e. has derivatives of all orders);

(ii) $\phi \geq 0$;

(iii) $\phi(x) = 0$ if $|x| \geq 1$;

(iv) $\int_{-\infty}^{\infty} \phi(x)dx > 0$.

4.18 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives. By considering the Taylor expansion of f, show that: $\|f'\|_\infty \leq \frac{1}{h}\|f\|_\infty| + h\|f''\|_\infty$ for every $h > 0$. By minimizing over h, show that $\|f'\|_\infty \leq 2\sqrt{\|f\|_\infty\|f''\|_\infty}$, where $\|g\|_\infty$ denotes $\sup_{x\in\mathbb{R}} |g(x)|$.

4.19 Define the Hermite polynomial of degree n by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} (\frac{d}{dx})^n e^{-\frac{x^2}{2}} \quad (n \geq 0, x \in I\!R).$$

Use Taylor's theorem to prove the identity

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{tx - \frac{1}{2}t^2}.$$

4.20 Let f be a real-valued differentiable function on an interval (a, b). Show that f is Lipschitz continuous if and only if f has bounded derivative.

5. Functions of bounded variation

5.1 If f is continuous on an interval $[a,b]$ with a bounded derivative in (a,b), show that f is of bounded variation on $[a,b]$. Is the boundedness of f' necessary for f to be of bounded variation? Justify your answer.

5.2 (a) Prove that if a real-valued function f is of bounded variation on an interval $[a,b]$, then f has right and left-hand limits at all $x \in (a,b)$.

(b) Prove that a function $f : [a,b] \to \mathbb{R}$ of bounded variation has at most countably many points of discontinuity.

5.3 Let $f(x) = x^2 \sin(\frac{1}{x}), g(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$ and $f(x) = g(x) = 0$ for $x = 0$. Show:

(a) f and g are differentiable everywhere (including $x = 0$),

(b) f is bounded variation on the interval $[-1,1]$, but g is not.

(c) Let $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Is f of bounded variation on $[-1,1]$?

6. Riemann integral, Lebesgue integral (via completion), convergence theorems

6.1 Let f be a positive function on $(0,1]$ such that f is Riemann integrable on $[t,1]$ for all $t \in (0,1)$, but $\lim_{x\to 0^+} f(x) = \infty$. Assume that the improper (Riemann) integral $(R)\int_0^1 f(x)dx$ exists. Show that f is a measurable function, Lebesgue integrable, and

$$\int_{[0,1]} f(x)dx \;=\; (R)\int_0^1 f(x)dx.$$

6.2 For each of the following problems, check whether the limit exists. If so, find its value.

$$(a)\ \lim_{n\to\infty}\int_1^n (1-\frac{x}{n})^n dx,\ (b)\ \lim_{n\to\infty}\int_1^{2n} (1-\frac{x}{n})^n dx.$$

6.3 (a) Characterize those bounded functions on $[0,1]$ which are Riemann integrable.

(b) Let (r_n) be an enumeration of the rationals in $[0,1]$. Define f on $[0,1]$ by

$$f(x) \;=\; \begin{cases} \frac{1}{n}, \text{ if } x = r_n \\ 0, \text{ if } x \text{ is irrational} \end{cases}$$

Is f Riemann integrable on [0,1]? Explain!

(c) Show that if

$$f(x) \;=\; \begin{cases} 1, \text{ if x is rational} \\ -1, \text{ if x is irrational} \end{cases}$$

then f is not Riemann integrable on the interval $[0,1]$. Is f Lebesgue integrable? Explain!

6.4 Show there are no bounded sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ for which $f_n(x) = a_n\sin(2\pi nx) + b_n\cos(2\pi nx)$ converges to 1 almost everywhere on $[0,1]$.

6.5 Let $f(x)$ be a real-valued measurable function on $[0,1]$. Show that

$$\lim_{n\to\infty}\int_0^1 (\cos(\pi f(x)))^{2n}\,dx = m\{x : f(x) \textit{ is an integer}\},$$

where m denotes Lebesgue measure.

6.6 (a) Show that $f(x) = x^{-r}$ is a Lebesgue integrable function on $[0,1]$ if $0 \leq r < 1$.

(b) If $0 \leq r < 1$ let $a_n = \displaystyle\int_0^1 \frac{dx}{n^{-1} + x^r}$ (Lebesgue integral). Compute $\lim_{n\to\infty} a_n$. Be sure to quote the appropriate integration theorems to justify your calculations.

6.7 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \dfrac{1}{n}, \text{ if } |x| \leq n \\ 0, \text{ if } |x| > n \end{cases}$$

(a) Show that f_n converges to 0 uniformly on R, and that $\lim_{n\to\infty}\int_{\boldsymbol{R}} f_n(x)\,dx = 2$ while $\int_{\boldsymbol{R}} (\lim_{n\to\infty} f_n)(x)\,dx \neq 2$.

(b) Explain why the example in part (a) does not contradict the Lebesgue dominated convergence theorem.

6.8 (a) Show that $f(x) = 1/\sqrt{x}$ is Lebesgue integrable on $(0,1)$.

(b) Find $\inf\{\int_0^1 \psi(x)\,dx | \psi \textit{ is a simple function, and } \psi(x) \geq 1/\sqrt{x} \textit{ on } (0,1)\}$. (Simple functions are finite linear combinations of characteristic functions of measurable sets with extended real-valued coefficients.)

6.9 Give an example of a sequence $(f_n)_{n\in \boldsymbol{N}}$ of bounded, measurable functions on $[0,1)$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0$$

but such that f_n converges pointwise nowhere.

6.10 Consider a Lebesgue-measurable function f or R with $\int\limits_{\boldsymbol{R}} f(t)^2\,dt < \infty$. show that the function $g(x) = \int\limits_{\boldsymbol{R}} f(t-x)f(t)\,dt$ is continuous.

6.11 Prove that $\lim_{n\to\infty} \int_{-\infty}^{\infty} (\sin nt) f(t) dt = 0$ for every Lebesgue integrable function f or $\mathbf{R}$. (Hint: Do the problem first for step functions.)

6.12 Let $f_n(x) = \dfrac{n}{x(lnx)^n}$ for $x \geq e$ and $n \in I\!N$.

(a) For which $n \in I\!N$ does the Lebesgue integral $\int\limits_e^{\infty} f_n(x)\,dx$ exist?

(b) Determine $\lim_{n\to\infty} f_n(x)$ for $x > e$.

(c) Does the sequence $(f_n)_{n\in \boldsymbol{N}}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

6.13 Define $f(x) = \int_{\boldsymbol{R}} \cos(xy)g(y)\,dy$ for $x \in I\!R$ where g is an integrable function on $I\!R$. Show that f is continuous.

6.14 Define $g : I\!R \to I\!R$ by $g(x) = 0$ if x is irrational and $g(x) = \frac{1}{q}$ if $x = p/q$ in lowest terms. Is g a Riemann integrable function? Give a proof of your assertion.

6.15 Give an example of a Lebesgue integrable function f on $[0,1]$ such that $\int_0^1 f(x)\,dx = 1$, but f is not Riemann integrable.

6.16 Let $f \in L^\infty[0,1]$ and $\int_0^1 x^n f(x)\,dx = 0$ for $n \in I\!N$. Show that $f = 0$ a.e.

6.17 Let f be a non-negative Lebesgue measurable function on $(0,\infty)$ such that f^2 is integrable. Let $F(x) = \int_0^x f(t)\,dt$ where $x > 0$. Show that $\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}} = 0$.

6.18 Let f be a differentiable function on $[-1,1]$. Prove that $\lim_{\epsilon\to 0} \int_{\epsilon<|x|\leq 1} \frac{1}{x} f(x)\,dx$ exists.

6.19 Let $f_n(x) := \dfrac{x^n}{n!} e^{-x}$ for $n \in I\!N_0$.

(a) Show that $\lim_{x\to\infty} f_n(x) = 0$ for all $x > 0$.

(b) Show that $f_n \in L^1(0,\infty)$ with $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}_0$.

(c) Show that $\lim_{k\to\infty} \int_0^k \frac{x^n}{n!}(1-\frac{x}{k})^k \, dx = 1$ for all $n \in \mathbb{N}_0$.

6.20 Prove that $\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx$ exists but that the function $\frac{\sin x}{x}$ is not integrable over $(0,\infty)$.

6.21 Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx$$

6.22 Let f be a continuous nonnegative function on $[a,b]$ where $a < b$. Let $M = \max\{f(x) : a \le x \le b\}$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n \, dx\right)^{\frac{1}{n}} = M.$$

6.23 Find and justify the limits: (a) $\displaystyle\lim_{n\to\infty} \int_0^n \frac{\sin x}{1+nx^2}\, dx$ and (b) $\displaystyle\lim_{n\to\infty} \int_0^{e^n} \frac{x}{1+nx^2}\, dx$.

6.24 Let $f_n(x) = \sum_{i=0}^{n-1} \frac{1}{n} f(x+\frac{i}{n})$, where f is a continuous function on $\mathbb{R}$. Show that the sequence of functions $(f_n)_{n\in\mathbf{N}}$ converges uniformly on every finite segment $[a,b]$ to the function $F(x) = \int_x^{x+1} f(s)\, ds$.

6.25 Let $f \in L^1(\mathbb{R})$ with respect to the Lebesgue measure such that $\int_{\mathbf{R}} |x||f(x)|\, dx < \infty$. Show that the function

$$g(y) = \int_{\mathbf{R}} e^{ixy} f(x) dx$$

is differentiable at every $y \in \mathbb{R}$.

6.26 Prove that if f is a real-valued Lebesgue integrable function on $\mathbb{R}$, then

$$\lim_{x\to 0} \int |f(x+t) - f(t)|\, dt = 0.$$

6.27 Let $f \in L^1(\mathbb{R})$.

(a) Prove: $\lim_{n\to\infty} \int_0^{1/n} f(x)dx = 0$.

(b) Prove or disprove: $\lim_{n\to\infty} \int_n^{\infty} f(x)\,dx = 0$.

6.28 Give an example of a sequence of uniformly bounded measurable functions f_n on $[0,1]$ such that $m\{x|f_n(x) \neq 0\} \to 0$ as $n \to \infty$, but the sequence $f_n(x)$ does not converge for any $x \in [0,1]$.

6.29 Let

$$f_n(x) = \begin{cases} n^2x, \text{ for } 0 \le x < \dfrac{1}{n} \\ 2n - n^2x, \text{ for } \dfrac{1}{n} \le x \le \dfrac{2}{n} \\ 0, \text{ for } \dfrac{2}{n} < x \le 1 \end{cases}$$

Sketch the graphs of f_1 and f_2. Prove that if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\,dx = 1$.)

6.30 Assume that the real valued measurable function $f(t,x)$ and its partial derivative $\frac{\partial}{\partial t} f(t,x)$ are bounded on $[0,1]^2$. Show that for $t \in (0,1)$

$$\frac{d}{dt}\left[\int_0^1 f(t,x)\,dx\right] = \int_0^1 \frac{\partial}{\partial t} f(t,x)\,dx.$$

Hint: Consider the difference quotient for the derivative on the left.

6.31 Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\,dx < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e., and

$$\int_0^1 \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x)dx.$$

6.32 Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\,dx = 0$.

6.33 Assume that $A \geq 0$, $B > 0$, and f continuous and nonnegative on $[a, b]$. Assume that $f(t) \leq A + B \int_a^t f(x)\,ds$ for $a \leq t \leq b$. Prove that $f(t) \leq Ae^{B(t-a)}$ for $a \leq t \leq b$.

7. Absolutely continuous functions and the fundamental theorem of calculus

7.1 Show that the product of two absolutely continuous functions on a closed finite interval $[a, b]$ is absolutely continuous.

7.2 (a) Show that a Lipschitz function is absolutely continuous.

(b) Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is bounded.

7.3 Let f be absolutely continuous in the interval $[\epsilon, 1]$ for each $\epsilon > 0$. Does the continuity of f at 0 imply that f is absolutely continuous on $[0, 1]$? What if f is also of bounded variation on $[0, 1]$?

7.4 A function $f : [0, 1] \to L^1[0, 1]$ is called Lipschitz continuous if there exists $M > 0$ such that $\|f(t) - f(s)\|_1 \leq M|t - s|$ for all $t, s \in [0, 1]$. It is called differentiable at a point $s \in (0, 1)$ if the differential quotients $\frac{f(t)-f(s)}{(t-s)}$ converge in $L^1[0, 1]$ as $t \to s$. Let $f : [0, 1] \to L^1[0, 1]$ be given by $f(t) = \chi_{[0,t]}$, where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0, t]$. Show that f is Lipschitz continuous and nowhere differentiable.

8. Basic properties of L^p-spaces, Riesz representation for L^p-spaces

8.1 Show that $(L_p[0,1], \|\cdot\|_p)$ is separable for $1 \le p < \infty$, but not separable for $p = \infty$.

8.2 Show that $L^p(0,1) \subset L^q(0,1)$ for any $p > q \ge 1$. Here the integrability is with respect to the Lebesgue measure. Is the inclusion map for $L^p(0,1)$ into $L^q(0,1)$ continuous?

8.3 Prove or disprove the equality $L^\infty[0,1] = \cap_{1 \le p < \infty} L^p[0,1]$.

8.4 Let $f \in L_p(\mathbb{R}), 1 \le p < \infty$. Show that $\displaystyle\int_{|x|>n} |f(x)|^p \, dx \to 0$ for $n \to \infty$.

8.5 Let $g_n = n\chi_{[0,n^{-3}]}$. Show that $\int_0^1 f(x) g_n(x)\, dx \to 0$ as $n \to \infty$ for all $f \in L^2[0,1]$, but not all $f \in L^1[0,1]$.

8.6 Construct an isometry of the Hilbert space ℓ_2 onto the Hilbert space $L_2\,[0,1]$ and justify that your map is an isometry.