Core-2 Exam
Graph Theory
Spring, 2002

Instructions. Solve any six from among the following eight problems. Submit only the six selected problems. You have 3 and 1/2 hours to complete this test. Good luck!

Note: Graphs are finite, undirected, and have no loops and no multiple edges.

1. Let G be a connected graph and let $\mathcal{T}$ be the set of all spanning trees of G. Define H to be the graph whose vertex set is $\mathcal{T}$ with two such vertices T_1 and T_2 connected by an edge whenever T_2 can be obtained from T_1 by replacing exactly one of the edges of T_1 with another edge of G. Is H necessarily connected?

2. Suppose G is a 3-regular graph that does not contain any cycles of length three, but does contain a cycle of odd length. What can you say about the chromatic number of G? What about the chromatic index of G?

3. Characterize graphs with no induced subgraph isomorphic to $K_{1,2}$.

4. Let G be a connected 3-regular plane graph such that every vertex of G is incident with two faces of length four and one face of length six. Use Euler's Formula to determine the number of vertices, edges, and faces of G. Draw G.

5. Prove that if the graph G is not complete and its minimum vertex degree is at least $(|V(G)| + k - 2)/2$, then G is k-connected.

6. Prove or disprove: If both a graph G and its complement are connected, then either G has only one vertex or contains an induced path on four vertices.

7. Prove or disprove: Every 2-connected 3-regular graph has a perfect matching.

8. Are the following two conditions equivalent for all graphs? Does one of them imply the other?
   a. G is Hamiltonian.
   b. For every subset S of $V(G)$, the graph $G - S$ has at most $|S|$ components.