Core-2 Exam
Graph Theory
Spring, 2002

Instructions. Solve any six from among the following eight problems. Submit only the six selected problems. You have 3 and 1/2 hours to complete this test. Good luck!

Note: Graphs are finite, undirected, and have no loops and no multiple edges.

1. Let $G$ be a connected graph and let $T$ be the set of all spanning trees of $G$. Define $H$ to be the graph whose vertex set is $T$ with two such vertices $T_1$ and $T_2$ connected by an edge whenever $T_2$ can be obtained from $T_1$ by replacing exactly one of the edges of $T_1$ with another edge of $G$. Is $H$ necessarily connected?

2. Suppose $G$ is a 3-regular graph that does not contain any cycles of length three, but does contain a cycle of odd length. What can you say about the chromatic number of $G$? What about the chromatic index of $G$?

3. Characterize graphs with no induced subgraph isomorphic to $K_{1,2}$.

4. Let $G$ be a connected 3-regular plane graph such that every vertex of $G$ is incident with two faces of length four and one face of length six. Use Euler’s Formula to determine the number of vertices, edges, and faces of $G$. Draw $G$.

5. Prove that if the graph $G$ is not complete and its minimum vertex degree is at least $(|V(G)| + k - 2)/2$, then $G$ is $k$-connected.

6. Prove or disprove: If both a graph $G$ and its complement are connected, then either $G$ has only one vertex or contains an induced path on four vertices.

7. Prove or disprove: Every 2-connected 3-regular graph has a perfect matching.

8. Are the following two conditions equivalent for all graphs? Does one of them imply the other?
   a. $G$ is Hamiltonian.
   b. For every subset $S$ of $V(G)$, the graph $G - S$ has at most $|S|$ components.