Algebra I Comprehensive Exam

August 22, 2008

Instructions: Do any five of the following six problems. You have three hours for this test. Good luck!

1. Let F be a field and let

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in F \right\}.$$

   (a) Verify that $H(F)$ is a nonabelian subgroup of $\mathrm{GL}(3,\, F)$.

   (b) If $|F| = q$, what is $|H(F)|$?

   (c) Find the order of all elements of $H(\mathbf{Z}/2\mathbf{Z})$.

   (d) Verify that $H(\mathbf{Z}/2\mathbf{Z}) \cong D_8$, the dihedral group of order 8.

2. Let R be an integral domain. Determine if each of the following statements about R-modules is true or false. Give a proof or counterexample, as appropriate.

   (a) A submodule of a free module is free.

   (b) A submodule of a free module is torsion-free.

   (c) A submodule of a cyclic module is cyclic.

   (d) A quotient module of a cyclic module is cyclic.

3. (a) Find all abelian groups of order 28.

   (b) Can a group of order 28 be simple? Prove your answer.

   (c) Exhibit two nonabelian groups of order 28 which are not isomorphic; justify your answer.

4. Let F be a field, and let V and W be vector spaces over F. Make V and W into $F[X]$-modules via linear operators T on V and S on W by defining $X \cdot v = T(v)$ for all $v \in V$ and $X \cdot w = S(w)$ for all $w \in W$. Denote the resulting $F[X]$-modules by V_T and W_S respectively.

   (a) Show that an $F[X]$-module homomorphism from V_T to W_S consists of an F-linear transformation $R : V \to W$ such that $RT = SR$.

   (b) Show that $V_T \cong W_S$ as $F[X]$-modules if and only if there is an F-linear isomorphism $P : V \to W$ such that $T = P^{-1}SP$.

5. (a) Prove that every Euclidean domain is a principal ideal domain (PID).

   (b) Give an example of a unique factorization domain that is not a PID and justify your answer.

6. Find all possible Jordan canonical forms for $A \in M_4(\mathbf{C})$ such that $A^3 = A^2$. In each case, give both the minimal and characteristic polynomial.