Instructions: *Do any five of the six problems below. Turn in ONLY the five problems you want graded. Write your name and the problem number clearly at the top of each page you turn in for grading. You have three hours.* Good luck!

1. First, give the **definition** of what it means for a finite group G to be **solvable**. Then prove that any group of order 21 is solvable. [Do not quote the Feit-Thompson theorem which states that all odd-order groups are solvable.]

2. Let G be a finite group and suppose G has a subgroup H of index p, where p is the smallest prime dividing the order of G. Prove that H is normal in G.

3. This problem has four parts. Let I and J be ideals in a commutative ring R with 1.

   a) Give the definition of $I + J$.

   b) Give the definition of IJ.

   c) Define what it means for the ideals I and J to be comaximal.

   d) Let the ideals I and J be comaximal. Prove that $IJ = I \cap J$.

4. Let $\mathbb{R}$ denote the field of real numbers, and let S denote the ring $S = M_2(\mathbb{R})$ of all 2×2-matrices with entries in $\mathbb{R}$. Prove that the zero ideal is a maximal ideal in S. Is $S/0$ a field?

5. Let $\mathbb{C}$ denote the field of complex numbers and let $T : \mathbb{C}^5 \longrightarrow \mathbb{C}^5$ be a linear transformation with $T^3 = -T$. Find all choices for T up to similarity.

6. Prove Cauchy's theorem: If the prime number p divides the order of a finite group G, then G contains an element of order p.