Algebra **Comprehensive/Qualifying Exam** **August 2010**

Directions: Do five of the following six problems. Start each problem on a new sheet of paper and put your name and the problem number at the top of every sheet. Hand in only the five problems you want graded. Good luck!

1. Let G be a group and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian. Conclude (with explanation) that then $Z = G$.

2. Let H and N be subgroups of a group G with N normal in G.
   (a) Give the definition of HN as a subset of G.
   (b) Prove that $NH = HN$ and that HN is a subgroup of G.

3. Let R be a commutative ring with identity. If $I \subseteq R$ is an ideal, then the radical of I, denoted $\sqrt{I}$, is defined by
$$\sqrt{I} = \{a \in R \ : \ a^n \in I \text{ for some positive integer n}\}.$$
   (a) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
   (b) If P is a prime ideal of R and r is a positive integer, find $\sqrt{P^r}$ and justify your answer.
   (c) Find $\sqrt{I}$, where I is the principal ideal $108Z$ in the ring $R = Z$ of integers.

4. (a) Prove that every Euclidean domain is a principal ideal domain (PID).
   (b) Give an example of a unique factorization domain that is not a PID and justify your answer.

5. Let R be an integral domain and let M be an R-module.
   (a) Define what it means to say that M is a free R-module.
   (b) Define what it means for a nonzero element of M to be a torsion element.
   (c) Prove that if M is a free R-module then M is torsion-free.

6. Let R be a commutative ring with 1 and let M be an R-module.
   (a) Define what it means for M to be R-cyclic.
   (b) Prove that M is R-cyclic if and only if $M \cong R/I$ for some ideal I in R.