Instructions: Do five of the six problems. Turn in only the five problems you want graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. Start each problem on a new sheet of paper. You have three hours. Good luck!

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1. Find all nonisomorphic abelian groups of order $2^4 \cdot 3^3$ which have no elements of order 2^3 and no elements of order 3^3. For each group, list the elementary divisors and the invariant factors.

2. Factor the polynomial
$$x^3 - 2x^2 + 4x + 2$$
into irreducible polynomials in each of the rings
(a) $Z_3[x]$ and
(b) $Z_5[x]$.

3. Let R be a commutative ring and n be a positive integer. Then $R[x^4]$ is an R-submodule of $R[x]$. Show that
$$R[x]/R[x^4] \cong R[x] \oplus R[x] \oplus R[x]$$
as R-modules.

4. Let R be an integral domain.
(a) State the definition of prime and irreducible elements of R.
(b) Prove that each prime element of R is irreducible.
(c) Prove that, if R is a principle ideal domain, then each irreducible element of R is prime.

5. Let N be a normal subgroup of a group G such that G/N is abelian. Prove that every subgroup K of G containing N is normal.

6. Let F be a field. The ring of formal power series $F[[x]]$ consists of all elements of the form
$$a_0 + a_1x + a_2x^2 + \ldots, \quad a_0, a_1, a_2 \in F$$
with addition and multiplication defined in an analogous way to the operations in a polynomial ring.
(a) Prove that the set of units of $F[[x]]$ consists of all elements
$$a_0 + a_1x + a_2x^2 + \ldots \in F[[x]]$$
such that $a_0 \neq 0$.
(b) Prove that all nonzero ideals of $F[[x]]$ are cyclic with generators x^n for some $n = 0, 1, \ldots$ (*Hint*: For a given nonzero ideal I of $F[[x]]$ consider the minimal n such that $x^n \in I$. Use part (a) to prove that $I = (x^n)$.)
(c) Conclude that for each field F, the ring $F[[x]]$ is a unique factorization domain.
