Instructions: Do ***five (5)*** *of the 8 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have 3 hours. *Good luck!*

Part A

1. Let G be a group of order $2p$ where p is an odd prime. If G has a normal subgroup of order 2, show that G is cyclic.

2. Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. Let $D \subseteq G$ be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbb{R}$ is arbitrary.

   (a) Show that U is a normal subgroup of G and that G/U is isomorphic to D.

   (b) True or False (with justification): $G \cong U \times D$.

Part B

3. Let $\mathbb{F}$ be a field and let $R = \mathbb{F}[X, Y]$ be the ring of polynomials in X and Y with coefficients from $\mathbb{F}$.

   (a) Show that $M = \langle X + 1, Y - 2\rangle$ is a maximal ideal of R.

   (b) Show that $P = \langle X + Y + 1\rangle$ is a prime ideal of R.

   (c) Is P a maximal ideal of R? Justify your answer.

4. Let $R = \mathbb{Z}[X]$. Answer the following questions about the ring R. You may quote an appropriate theorem, provide a counterexample, or give a short proof to justify your answer.

   (a) Is R a unique factorization domain?

   (b) Is R a principal ideal domain?

   (c) Find the group of units of R.

   (d) Find a prime ideal of R which is not maximal.

   (e) Find a maximal ideal of R.

5. Let R be an integral domain. Determine if each of the following statements about R-modules is true or false. Give a proof or counterexample, as appropriate.
   (a) A submodule of a free module is free.
   (b) A submodule of a free module is torsion-free.
   (c) A submodule of a cyclic module is cyclic.
   (d) A quotient module of a cyclic module is cyclic.

Part C

6. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces V and W. Show that $\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$.

7. Let $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the linear transformation expressed relative to the standard basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ by the matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

   (a) Find a basis for $\operatorname{Ker}(T)$.
   (b) Find a basis for $\operatorname{Im}(T)$.
   (c) Find the matrix for T expressed in the basis $\mathbf{f}_1 = (-1, 1, 0)$, $\mathbf{f}_2 = (0, 1, -1)$, $\mathbf{f}_3 = (1, 0, 1)$.

8. Let S and T be linear transformations between finite-dimensional vector spaces V and W over the field $\mathbb{F}$. Show that $\operatorname{Ker} S = \operatorname{Ker} T$ if and only if there is an invertible operator U on W such that $S = UT$.