Instructions: Complete any **five (5)** of the following problems. Turn in only these five problems to be graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

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1. Suppose that G is a finite group and H is a subgroup, with $|G| = g$ and $|H| = h$ elements. Lagrange's Theorem states that h divides g.
   (a) Give a brief proof of Lagrange's Theorem.
   (b) Prove that when G is cyclic, the converse of Lagrange's Theorem also holds: if $n \mid g$, then there is a subgroup H with n elements.
   (c) If G is an abelian group with 40 elements, must it have a subgroup with 10 elements?

2. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is prime. Let $S = \mathrm{SL}_2(\mathbb{F}_p)$ be the subgroup of matrices with determinant 1.
   (a) Determine the group orders $|G|$ and $|S|$.
   (b) Prove that S is a normal subgroup of G.
   (c) Determine the quotient group G/S.

3. An element a in a ring R is *nilpotent* if $a^n = 0$ for some natural number n.
   (a) If R is a commutative ring with identity, show that the set of nilpotent elements forms an ideal.
   (b) Describe all of the nilpotent elements in the ring $\mathbb{R}[x]/(f(x))$, where $(f(x))$ is the ideal generated by
   $$f(x) = \left(x^2 - 1\right)\left(x^3 - 1\right).$$

4. (a) Let R be a ring with units $R^\times$, and let $M := R \setminus R^\times$, the set of all non-units. Prove that if M is an ideal, then it is the unique maximal ideal in R.
   (b) If F is a field, find a maximal ideal in the polynomial ring $F[x]$.

5. Let M be an R-module and let $f : M \to M$ be an R-module endomorphism that is idempotent, that is, $f \circ f = f$. Prove that $M \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$.

6. Suppose that V is a vector space over the field F, and that $T : V \to V$ is an F-linear transformation.
   (a) Prove that V is an $F[x]$-module under the action
   $$p(x).v = \left(a_n x^n + \cdots + a_1 x + a_0\right).v = a_n T^n(v) + \cdots + a_1 T(v) + a_0 v.$$
   (b) The *annihilator* of V is the ideal $A \subset F[x]$ such that $a.v = 0$ for all $a \in A, v \in V$. Determine the annihilator if $V = \mathbb{R}^3 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ and the action of T is the cyclic rotation $T(e_1) = e_2, T(e_2) = e_3, T(e_3) = e_1$. (*Hint: The annihilator is related to the minimal polynomial of T.*)