Instructions: Complete any **five (5)** of the following problems. Turn in only these five problems to be graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

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1. Two subsets S, T in a group G are *conjugate* if there is some $g \in G$ such that

$$S = gTg^{-1} = \{gtg^{-1} \mid t \in T\}.$$

   (a) Prove that if H is a subgroup of G, then any conjugate gHg^{-1} is also a subgroup of G.

   (b) Let $\sigma = (12)(34), \tau = (13)(25)$ be elements of the symmetric group S_5. Determine whether or not σ and τ are conjugate, and justify your answer.

2. (a) Prove that the set of all elements of finite order in an abelian group form a subgroup.

   (b) Show that there is a nonabelian group such that the elements of finite order do **not** form a subgroup.

3. (a) Prove that every Euclidean domain is a principal ideal domain (PID).

   (b) Give an example of a unique factorization domain that is not a PID, and justify your answer.

4. Suppose that R is an integral domain and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x]$ is a monic polynomial. Eisenstein's Irreducibility Criterion states that if $p \in R$ is a prime such that $p \mid a_i$ for all $0 \le i \le n-1$, and $p^2 \nmid a_0$, then $f(x)$ is irreducible in $R[x]$.

   (a) Prove Eisenstein's Criterion.

   (b) Let $f(x) = x^4 + 4x^3 + 2x^2 - 2 \in R[x]$. Factor $f(x)$ into a product of irreducibles for $R = \mathbb{Z}$ and $R = \mathbb{Z}/5\mathbb{Z}$.

5. Let G be an abelian group and K a subgroup of G. For each of the following statements, decide if it is true or false. Give a proof or provide a counterexample, as appropriate.

   (a) If $G/K \cong \mathbb{Z}^2$, then $G \cong K \oplus \mathbb{Z}^2$.

   (b) If $G/K \cong \mathbb{Z}/2\mathbb{Z}$, then $G \cong K \oplus \mathbb{Z}/2\mathbb{Z}$.

6. Suppose that R is a commutative ring and M is an R-module. The *annihilator* of M is

$$\mathrm{Ann}(M) := \{r \in R \mid r.m = 0 \;\forall\, m \in M\}.$$

   (a) Prove that $\mathrm{Ann}(M)$ is an ideal of R.

   (b) Let M be an $\mathbb{R}[x]$-module. Suppose that the action of x on M corresponds (with a certain choice of basis) to the linear transformation

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 5 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Determine $\mathrm{Ann}(M)$.