Instructions: Complete any **five (5)** of the following problems. Turn in **only** these five problems to be graded. Write your name and the problem number at the top of each page that you turn in for grading. You have three hours. Good luck!

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1. (a) Suppose that G be a finite group of order n. Prove Cayley's Theorem: There is a subgroup $H < S_n$ such that G is isomorphic to H.

   (b) Let D_8 be the dihedral group of order 8. Show that D_8 is isomorphic to a subgroup of S_4.

2. Let $G_p = \mathrm{GL}_2(\mathbb{F}_p)$ be the multiplicative group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is prime.

   (a) Write down all the elements of G_2.

   (b) Determine the group order $|G_p|$ for general p.

   (c) Prove that the center of G_p is $Z(G_p) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{F}_p \setminus \{0\} \right\}$.

3. Let $F := \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Q} \right\}$.

   (a) Prove that F is a field under the usual matrix operations of addition and multiplication.

   (b) Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \right\}$.

4. Let $R := \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients.

   (a) Show that $M = (2, x)$ is a maximal ideal of R.

   (b) Show that $P = (x - 1)$ is a prime ideal of R.

   (c) Is P also a maximal ideal?

5. Let M be the $\mathbb{Z}$-submodule of $\mathbb{Z}^2$ generated by $(4, 6)$, and let $H := \mathbb{Z}^2/M$ be the quotient module.

   (a) Identify the rank and invariant factors of H; in other words, find the $r \in \mathbb{Z}_{\geq 0}$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that

   $$H \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}.$$

   (b) Let M_1 be the $\mathbb{Z}$-submodule of $\mathbb{Z}^2$ generated by $(3, 6)$, and let M_2 be generated by $(2, 2)$. Determine which of the quotients $\mathbb{Z}^2/M_1$ and $\mathbb{Z}^2/M_2$ is isomorphic to H.

6. Suppose that R is a commutative ring with 1. Let M be an R-module, and let N be an R-submodule of M. Show that if N and M/N are finitely generated, then M is finitely generated.