Instructions: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Find a Sylow 2-subgroup of S_4 and identify its isomorphism type.
   (b) How many Sylow 2-subgroups of S_4 are there? Please justify your answer.

2. Let G be a finite abelian group and n an integer relatively prime to $|G|$. Prove that the function $\varphi : G \to G$ defined by $\varphi(g) = g^n$ is an isomorphism of groups.

Part B

3. Consider the polynomial $f(X) = X^3 + 2X^2 - X + 4$.
   (a) Prove or disprove that $\mathbb{Z}_3[X]/(f(X))$ is a field.
   (b) Prove or disprove that $\mathbb{Z}_5[X]/(f(X))$ is a field.

4. Let R be a commutative ring with identity. Suppose I, J are ideals of R such that $I+J = R$.
   (a) Prove that $IJ = I \cap J$.
   (b) Prove that $I^n + J^m = R$ for all positive integers m, n.

5. Let $\mathbb{F}$ be a field. Prove that the center of the matrix ring $M_n(\mathbb{F})$ is the set of scalar matrices $\{aI \mid a \in \mathbb{F}\}$.

Part C

6. Prove or disprove the following statements:
   (a) If $M \subset \mathbb{Z}^n$ is a $\mathbb{Z}$-submodule of rank n, then $\mathbb{Z}^n/M$ is a finite group.
   (b) If G is a finite abelian group and M is a subgroup of G such that $G/M \cong \mathbb{Z}_3$, then $G \cong M \oplus \mathbb{Z}_3$.

7. Consider the $\mathbb{Q}[X]$-module V with V being the column vector space $\mathbb{Q}^3$ and the action of X given by $X \cdot v := A \cdot v$ for $v \in V$, where
$$A = \begin{bmatrix} 0 & 3 & -1 \\ -1 & -4 & 1 \\ -2 & -6 & 1 \end{bmatrix} \in M_3(\mathbb{Q})\,.$$
   (a) Find the invariant factors of the $\mathbb{Q}[X]$-module V and its invariant factor decomposition up to isomorphism.
   (b) Determine the Jordan canonical form of A from (a).
