Instructions: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. Let $G = SL(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$ and determinant 1, where p is a prime.
   (a) Show that G has order $p^3 - p$.
   (b) Show that for $p = 2$ the group G is isomorphic to the symmetric group S_3.

2. Let p be an odd prime. Classify all groups of order $2p$.

Part B

3. (a) Show that the ring $\mathbb{Q}[X]/(X^2+13)$ is isomorphic to $\mathbb{Q}[\sqrt{-13}] = \{a+b\sqrt{-13} : a, b \in \mathbb{Q}\}$.
   (b) Is $\mathbb{Z}[\sqrt{-13}]$ a principal ideal domain? Please justify your answer.
   (c) Is $\mathbb{Z}[\sqrt{-13}]$ a Euclidean domain? Please justify your answer.

4. (a) Let p be a prime. Is $\mathbb{Q}[X]/(X^{p-1} + X^{p-2} + \cdots + X + 1)$ a field? Please justify.
   (b) Let $\mathbb{F}_{13} = \mathbb{Z}/13\mathbb{Z}$ be the finite field of size 13. Is $\mathbb{F}_{13}[X]/(X^2+1)$ a field? Please justify.

5. Let R be a ring and $R^\times$ be the set of units of R.
   (a) Let $M = R \setminus R^\times$. If M is an ideal, prove that M is a maximal ideal and that moreover it is the only maximal ideal of R.
   (b) Give an example of such a ring that is not a field nor a division ring.

Part C

6. (a) Prove that $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, the group of $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$, is isomorphic to $\mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.
   (b) Let G be an abelian group and K a subgroup. Decide if it is true or false. Give a proof or provide a counterexample, as appropriate. If $G/K \cong \mathbb{Z}/2\mathbb{Z}$, then $G \cong K \oplus \mathbb{Z}/2\mathbb{Z}$.

7. Consider the $\mathbb{Q}[X]$-module V with V being the column vector space $\mathbb{Q}^4$ and the action of X given by $X \cdot v := A \cdot v$ for $v \in V$, where
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{bmatrix} \in M_4(\mathbb{Q})\,.$$
   (a) Find the invariant factors of the $\mathbb{Q}[X]$-module V and its invariant factor decomposition up to isomorphism.
   (b) Consider $A \in M_4(\mathbb{C})$ and find the Jordan canonical form of A.