Instructions:

1. Do 5 of the following problems, including at least one from Part A, one from Part B, and one from Part C. **Please justify your answers**.

2. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading.

You have three hours. Good luck!

Part A

1. Let p be a prime. Classify all groups of order $2p$.

2. (a) Prove *Burnside's Formula*: Let G be a finite group and A a finite set. If G acts on A and r is the number of orbits in A under G, show that

$$r = \frac{1}{|G|} \sum_{g \in G} |A_g|,$$

where $A_g := \{a \in A : \ g \cdot a = a\}$.

(b) Find the number of distinct ways to paint the edges of a square if the number of available colors is n, *the same color can be used on any number of edges, and two colorings are considered the same if one can be obtained from the other by rotation or reflection.*

Part B

3. (a) Let R be a PID and let I, J be nonzero ideals of R. Show that $IJ = I \cap J$ if and only if $I + J = R$.

(b) Prove or disprove that $\mathbb{Z}/900\mathbb{Z}$ is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$ as rings.

4. (a) Let $\mathbb{F}$ be any field and $\mathbb{F}^*$ be the multiplicative group of $\mathbb{F}$. Prove or disprove that every finite subgroup of $\mathbb{F}^*$ is cyclic.

(b) Let H be any finite subgroup of $\mathbb{C}^*$. Prove or disprove that $\mathbb{C}^*/H$ is isomorphic to $\mathbb{C}^*$.

5. Let p be an prime and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Find all prime numbers p such that $\mathbb{F}_p[x]/ < x^2 + 1 >$ is a field.

Part C

6. (a) Find a basis and the invariant factors of the submodule N of $\mathbb{Z} \times \mathbb{Z}$ generated by $(-6, 2)$, $(2, -2)$, and $(10, 6)$.

   (b) Let p be a prime. Find the number of distinct subgroups of index p in $\mathbb{Z} \times \mathbb{Z}$.

7. Let $\mathbb{F}$ be a field and let V and W be vector spaces over $\mathbb{F}$. Make V and W into $\mathbb{F}[x]$-modules via linear operators $T \in \mathrm{End(V)}$ and $S \in \mathrm{End(W)}$ by defining

$$x \cdot v = T(v),\ \forall v \in V$$
$$x \cdot w = S(w),\ \forall w \in W.$$

   Denote the resulting $\mathbb{F}[x]$-modules by V_T and W_S respectively.

   (a) Show that an $\mathbb{F}[x]$-module homomorphism from V_T to W_S consists of an $\mathbb{F}$-linear isomorphism $R\colon V \longrightarrow W$ such that $RT = SR$.

   (b) Show that $V_T \cong W_S$ as $\mathbb{F}[x]$-modules if and only if there is an $\mathbb{F}$-linear isomorphism $R\colon V \longrightarrow W$ such that $T = R^{-1}SR$.