Algebra Qualifying Exam, August 2022

Do five of the following problems, including at least one from each of parts A, B, and C.

Part A

1. Let G be a group and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian.
   (c) Let D_6 be the dihedral group of order 6. Find the center of D_6
2. Let p, q, and r be distinct prime numbers.
   (a) Show that if G is a group of order pqr, then G is not simple.
   (b) Show that G is solvable.
3. Let $G = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \mid a, b \in \mathbb{R}, a > 0 \right\}$ and $N = \{ \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix} \right) \mid c \in \mathbb{R} \}$.
   (a) Show that N is a normal subgroup of G, and G/N is isomorphic to the multiplicative group $\mathbb{R}_+$ of positive real numbers.
   (b) Find a group N' with $N \lneq N' \lneq G$ or prove that no such N' exists.

Part B

4. (a) Let R be an integral domain. Show that the group of units of the polynomial ring $R[X]$ is equal to the group of units of the ground ring R.
   (b) Show that this is not true for $R = \mathbb{Z}/4\mathbb{Z}$.
5. Let $R_1 = \mathbb{F}_p[X]/(X^2 - 2)$ and $R_2 = \mathbb{F}_p[X]/(X^2 - 3)$. Determine whether R_1 is isomorphic to R_2 in each of the cases $p = 2$, $p = 5$, and $p = 11$.
6. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

Part C

7. Let M be an R-module, and let $f : M \to M$ be an idempotent endomorphism. Prove that $M \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$.
8. Let V be a $\mathbb{Q}[x]$-module corresponding to a matrix $A \in M_n(\mathbb{Q})$. Prove that V is cyclic if and only if the ideal $\mathrm{Ann}(V)$ is generated by the characteristic polynomial of A.