Do five of the following problems, including at least one from each of parts A, B, and C.

Part A

1. Let $G$ be a group and let $Z$ denote the center of $G$.
   
   (a) Show that $Z$ is a normal subgroup of $G$.
   
   (b) Show that if $G/Z$ is cyclic, then $G$ must be abelian.
   
   (c) Let $D_6$ be the dihedral group of order 6. Find the center of $D_6$

2. Let $p$, $q$, and $r$ be distinct prime numbers.
   
   (a) Show that if $G$ is a group of order $pqr$, then $G$ is not simple.
   
   (b) Show that $G$ is solvable.

3. Let $G = \{(a \ b \ 0 \ 0) | a, b \in \mathbb{R}, a > 0\}$ and $N = \{(1 \ 0 \ 0 \ c) | c \in \mathbb{R}\}$.
   
   (a) Show that $N$ is a normal subgroup of $G$, and $G/N$ is isomorphic to the multiplicative group $\mathbb{R}_+$ of positive real numbers.
   
   (b) Find a group $N'$ with $N \leq N' \leq G$ or prove that no such $N'$ exists.

Part B

4. (a) Let $R$ be an integral domain. Show that the group of units of the polynomial ring $R[X]$ is equal to the group of units of the ground ring $R$.
   
   (b) Show that this is not true for $R = \mathbb{Z}/4\mathbb{Z}$.

5. Let $R_1 = \mathbb{F}_p[X]/(X^2 - 2)$ and $R_2 = \mathbb{F}_p[X]/(X^2 - 3)$. Determine whether $R_1$ is isomorphic to $R_2$ in each of the cases $p = 2$, $p = 5$, and $p = 11$.

6. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

Part C

7. Let $M$ be an $R$-module, and let $f : M \to M$ be an idempotent endomorphism. Prove that $M \cong \text{Ker}(f) \oplus \text{Im}(f)$.

8. Let $V$ be a $\mathbb{Q}[x]$-module corresponding to a matrix $A \in M_n(\mathbb{Q})$. Prove that $V$ is cyclic if and only if the ideal $\text{Ann}(V)$ is generated by the characteristic polynomial of $A$. 