ALGEBRA QUALIFYING EXAM

August 2023

Answer five of the following ten questions, including at least one from each of parts I, II, and III.

Part I

(1) Let $G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \,\middle|\, a, b \in \mathbb{R},\ a > 0 \right\}$ and $N = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$. These are groups under matrix multiplication.

(a) Show that N is a normal subgroup of G, and that G/N is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}^+$.

(b) Does there exist a group H such that $N \subsetneq H \subsetneq G$? Prove your answer.

(2) Let G be a finite group, and suppose there is an automorphism $\sigma : G \to G$ with the following two properties: (i) $\sigma(g) = g$ if and only if $g = 1$, and (ii) $\sigma \circ \sigma$ is the identity map $G \to G$.

(a) Show that every element of G can be written in the form $x^{-1}\sigma(x)$.

(b) Show that G is abelian. (*Hint:* Apply σ to the expression from part (a).)

(3) Let G be a group of order 105. Show that if G contains a normal Sylow 3-subgroup P, then P is contained in the center of G.

Part II

(4) Let $\mathbb{F}$ be a field, and let $R = \mathbb{F}[x, y]$ be the ring of polynomials in two variables with coefficients in $\mathbb{F}$.

(a) Show that $M = \langle x - 3, y + 1 \rangle$ is a maximal ideal of R.

(b) Show that $P = \langle x^2 - y \rangle$ is a prime ideal of R.

(c) Is P a maximal ideal of R? Justify your answer.

(5) Let I be an ideal in a commutative ring R. Let

$$\text{rad } I = \{ r \in R \mid r^n \in I \text{ for some integer } n \geq 1 \}.$$

Prove that rad I is an ideal.

(6) Let R be an integral domain that contains a field k as a subring. Then R can be regarded as a k-vector space. Assume that as a k-vector space, R is finite-dimensional. Prove that R is a field.

(7) Show that the ring $\mathbb{Z}[x]$ is *not* a principal ideal domain.

Part III

(8) Let $\mathbb{Z}[\frac{1}{2}]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Is $\mathbb{Z}[\frac{1}{2}]$ finitely generated as a $\mathbb{Z}$-module?

(9) Let R be a ring, and let M be an R-module. Let $f : M \to M$ be an R-module homomorphism such that $f \circ f = f$. Prove that $M \cong \ker(f) \oplus \text{im}(f)$.

(10) Prove that there is no 2×2 matrix A with entries in $\mathbb{C}$ satisfying $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.