ALGEBRA QUALIFYING EXAM, AUGUST 2025

Do 5 of the following problems, including at least one from each of parts A, B, and C. The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. Let S_3 be the group of permutations of 3 elements.
   (a) Find all the subgroups of S_3. For each subgroup H say whether H is normal or not.
   (b) Show that S_3 is isomorphic with $\mathrm{GL}(2, \mathbf{Z}/2)$, the group of 2 by 2 invertible matrices with entries in the integers modulo 2. Here it might be useful to show that S_3 has a presentation

$$\langle a, b \mid a^3 = b^2 = e, bab = a^2 \rangle$$

2. Let G be a group with 20 elements.
   a) Show that G has a unique subgroup N of order 5, and that N is a normal subgroup. What are the possibilities for G/N?
   b) Give an example of a nonabelian group of order 20.
3. Let G be the group of upper triangular matrices of determinant 1:

$$G = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}, \quad a \neq 0, b \in \mathbb{R}.$$

   a) Show that the subset $N \subset G$ with $a = 1$ is a subgroup, and in fact a normal subgroup.
   b) Show that

$$\psi\left(\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\right) = a$$

   determines an isomorphism from G/N to the group $(\mathbb{R}^*, \times)$.

Part B

1. Let $I, J \subset \mathbb{Z}[X]$ be the ideals $I = \langle 5, X^2 + 1\rangle$, $J = \langle 5, X^2 + X + 1\rangle$.
   a) Show that there is a ring isomorphism

$$\mathbb{Z}[X]/I \cong \mathbb{F}_5[X]/\langle X^2 + 1\rangle \cong \mathbb{F}_5 \times \mathbb{F}_5.$$

   Is I a prime ideal? Explain. Hint: Factor $X^2 + 1$ mod 5.
   b) Show that $X^2 + X + 1$ is irreducible mod 5. Explain why this shows that $\langle X^2 + X + 1\rangle$ is a maximal ideal in $\mathbb{F}_5[X]$. Then explain why $\mathbb{Z}[X]/J$ is a field. How many elements does it have?
2. a) Find all the ideals in the ring $\mathbb{Z}/36$.
   b) Find all the ideals in the ring $\mathbb{Q}[X]/\langle X^3 - 3X - 2\rangle$
3. Let $R = \mathbb{Z}[X]$.
   a) Give an example of an ideal in R that is not principal. Explain your example.
   b) Give an example of a nonzero prime ideal P in R. Describe R/P.
   c) Give an example of a maximal ideal M in R. Describe R/M.

Part C

1. Let
$$A = \begin{pmatrix} 2 & -4 & 1 & 3 \\ 2 & -3 & 0 & 2 \\ 0 & -1 & 1 & 2 \\ 1 & -1 & -1 & 0 \end{pmatrix}.$$
The Smith form of $xI_4 - A$ is $\mathrm{diag}(1, 1, x^2+1, x^2+1)$.
   a) Write down the rational canonical form C of A, and the Jordan canonical form J of A.
   b) Are there matrices $P, Q \in GL_4(\mathbb{Q})$ such that $C = PAP^{-1}$ and $J = QAQ^{-1}$? Explain.
2. Let
$$T = \begin{pmatrix} 6 & 6 & 24 \\ 2 & 6 & 20 \\ 4 & 18 & 64 \end{pmatrix}.$$
This defines a homomorphism of $\mathbb{Z}$-modules
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto T \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbb{Z}^3 \xrightarrow{T} \mathbb{Z}^3.$$
   a) Using row and column operations over the ring $\mathbb{Z}$ bring T into Smith normal form.
   b) In terms of the Structure Theorem for Finitely Generated Abelian Groups, describe $\mathbb{Z}^3/\mathrm{Im}(T)$.
3. Let R be a commutative ring.
   a) Define what it means for an R-module M to be Noetherian.
   b) State Hilbert's Basis Theorem.