

Algebra Qualifying Exam

August 2026

Do **5** of the following problems, including at least one from each of parts A, B, and C. The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have **three** hours. Good luck!

Part A

- (1) Let $G = S_4$ be the symmetric group on four letters.
 - (a) Determine the number of Sylow 2-subgroups and the number of Sylow 3-subgroups of G . Justify your answers.
 - (b) Identify, up to isomorphism, a Sylow 2-subgroup of G .
- (2) Let G be a group and let $Z = Z(G)$ denote its center.
 - (a) Show that Z is a normal subgroup of G .
 - (b) Show that if the quotient G/Z is cyclic, then G is abelian.
 - (c) Conclude that every group of order p^2 , where p is a prime, is abelian.
- (3) Let p be an odd prime and let G be a group of order $2p$.
 - (a) Show that G has a unique (hence normal) subgroup of order p .
 - (b) Prove that G is isomorphic to either the cyclic group C_{2p} or the dihedral group D_p of order $2p$.

Part B

- (4) Let $\mathbb{F}_2$ be the field with two elements and let $f(X) = X^3 + X + 1 \in \mathbb{F}_2[X]$.
 - (a) Show that $f(X)$ is irreducible over $\mathbb{F}_2$.
 - (b) Let $K = \mathbb{F}_2[X]/\langle f(X) \rangle$. Determine $|K|$ and explain why K is a field.
 - (c) Show that the multiplicative group $K^\times$ is cyclic, and exhibit a generator.
- (5) Consider the polynomial ring $R = \mathbb{Z}[X]$.
 - (a) Show that R is not a principal ideal domain.
 - (b) Give an example of a prime ideal of R that is not maximal, and justify your answer.
 - (c) Give an example of a maximal ideal of R , and justify your answer.
- (6) Let $R = \mathbb{Z}[i] = \{a+bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers, with norm $N(a+bi) = a^2+b^2$.
 - (a) Prove that R is a Euclidean domain with respect to N .
 - (b) Determine the group of units of R .
 - (c) Show that 3 is irreducible in R , but 5 is not. Factor 5 into irreducibles in R .

Part C

- (7) Let G be the abelian group with generators x, y, z subject to the relations

$$5x + 9y + 5z = 0, \quad 2x + 4y + 2z = 0, \quad x + y - 3z = 0.$$

Determine the elementary divisors of G and write G as a direct sum of cyclic groups.

- (8) Let R be a ring. An R -module N is called *simple* if $N \neq 0$ and its only submodules are 0 and N .

(a) Prove *Schur's Lemma*: if $\varphi : N \rightarrow N'$ is a homomorphism of simple R -modules, then φ is either the zero map or an isomorphism.

(b) Deduce that $\text{End}_R(N)$ is a division ring for any simple module N .

- (9) Let V be the vector space of polynomials $p(X) \in \mathbb{C}[X]$ of degree ≤ 3 , and define a linear operator $T : V \rightarrow V$ by $T(p(X)) = p'(X)$, the derivative of p .

(a) Write the matrix of T with respect to the basis $\{1, X, X^2, X^3\}$.

(b) Compute the characteristic polynomial and the minimal polynomial of T .

(c) Find the Jordan canonical form of T .

- (10) Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements and let V be a 2-dimensional vector space over $\mathbb{F}_p$.

(a) Find the number of linear transformations $T : V \rightarrow V$.

(b) Find the number of invertible linear transformations $T : V \rightarrow V$.

(c) Find the number of linear transformations $T : V \rightarrow V$ that are nilpotent.