Algebra Qualifying Exam
January 8, Thursday, 1:00–4:00PM 2026

Do **5** of the following problems, including at least one from each of parts A, B, and C. The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have **three** hours. Good luck!

Part A

(1) (a) State the structure theorem for finitely generated abelian groups.

(b) If p and q are distinct primes, determine the number of nonisomorphic abelian groups of order p^3q^4.

(2) (a) Define what it means for a group G to act on a set A.

(b) The group $\mathrm{GL}_2(\mathbb{C})$ acts by left-multiplication on the set of matrices $M_{2,5}(\mathbb{C})$ (2 by 5 matrices with complex number entries). Describe the orbits. How many are there?

(3) Let G be an abelian group (maybe infinite). Suppose we have a real function $f : G \to [0, \infty)$ satisfying

(i) for any real number r, we have $\{g \in G \mid f(g) \leq r\}$ is a finite set,

(ii) there is a finite subset $S \subset G$ and a positive real number κ such that for any $g \in G$, there exists some $h \in G$ where $g - 2h \in S$ with $f(h) \leq \frac{1}{2}f(g) + \kappa$.

Prove that G is finitely generated by competing the following steps.

(a) Let $H = \langle S \rangle$ be the subgroup of G generated by S. Show inductively that for any $g \in G$ and $n \geq 1$, there is some $g_n \in G$ such that $g - 2^n g_n \in H$ and $f(g_n) \leq \frac{1}{2^n}f(g) + \kappa(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}})$.

(b) Show that for any $g \in G$, there is some $h \in G$ such that $g - 2^n h \in H$ and $f(h) \leq 3\kappa$.

(c) Show that G is generated by S and all the elements in $\{g \in G \mid f(g) \leq 3\kappa\}$.

Part B

(4) Let

$$F = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} \ : \ a, b \in \mathbb{Q} \right\}.$$

(a) Prove that F is a field under the usual matrix operations of addition and multiplication.

(b) Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{2})$.

(5) Let $f : R \to S$ be a ring homomorphism between two commutative rings R, S.

(a) Show that for any prime ideal P in S, its pre-image $f^{-1}(P) = \{r \in R \mid f(r) \in P\}$ is also a prime ideal in R.

(b) Prove or disprove whether part (a) still holds when we change "prime" into "maximal".

(6) Let R be an integral domain and K its fractional field. Suppose for any nonzero element $x \in K$, we have either $x \in R$ or $\frac{1}{x} \in R$.

(a) Show that for any nonzero elements $a, b \in R$, we have either $a \mid b$ or $b \mid a$.

(b) Show that R is a local ring. That means it has one unique maximal ideal which contains all the non units in R.

Part C

(7) Let R be a ring and M be a left R-module. Suppose $f : M \to M$ is an R-module endomorphism which is idempotent, that is, $f \circ f = f$. Prove that $M \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$ as R-modules.

(8) Let p be a prime number, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements, $V = \mathbb{F}_p^4$ (a 4-dimensional vector space over $\mathbb{F}_p$), and W the subspace of V spanned by the three vectors $\mathbf{a}_1 = (1, 2, 2, 1)$, $\mathbf{a}_2 = (0, 2, 0, 1)$, and $\mathbf{a}_3 = (-2, 0, -4, 3)$. Find $\dim_{\mathbb{F}_p} W$. (Note that this dimension depends on p.)

(9) Let V be a finite-dimensional complex vector space and $f : V \to V$ any linear operator on V. Choose a basis $\{v_1, \ldots, v_n\}$ for V and its dual basis $\{v^1, \ldots, v^n\}$ for V^* satisfying $v^i(v_j) = 1$ if $i = j$ and $v^i(v_j) = 0$ if $i \neq j$. Define the trace of f as a complex number given by $\mathrm{Tr}(f) = \sum_{i=1}^n v^i(f(v_i))$.

(a) Write $f(v_i) = \sum_{j=1}^n m_{ij} v_j$ for some square matrix $(m_{ij})_{n \times n}$. Show that $\mathrm{Tr}(f) = \sum_{i=1}^n m_{ii}$.

(b) Show that $\mathrm{Tr}(f) = \sum_{i=1}^n \alpha_i$ where α_i are the eigenvalues for f counting with multiplicity. Hence, conclude that $\mathrm{Tr}(f)$ does not depend on the choice of the basis $\{v_1, \ldots, v_n\}$.

(c) A sequence of complex numbers $\{a_0, a_1, \ldots\}$ is said to be linearly recursive if it satisfies a non-zero polynomial $g(x) = g_0 + g_1 x + \cdots + g_{m-1} x^{m-1} + g_m x^m \in \mathbb{C}[x]$ that is

$$g_0 a_n + g_1 a_{n+1} + \cdots + g_{m-1} a_{n+m-1} + g_m a_{n+m} = 0, \quad \text{for all } n \geq 0.$$

Show that the sequence $\{\mathrm{Tr}(f^i) \mid i = 0, 1, \ldots\}$ satisfies the characteristic polynomial of the linear operator f and hence is linearly recursive.

(10) Let V, W be two complex vector spaces.

(a) Show that there is a well-defined linear map $\psi : V^* \otimes W \to \mathrm{Hom}_{\mathbb{C}}(V, W)$ given by $\psi(f \otimes w)(v) = f(v)w$ for any $v \in V, w \in W$ and linear functional $f \in V^*$.

(b) If $\dim V < \infty$, show that ψ is both injective and surjective. Hence, ψ induces a vector space isomorphism $V^* \otimes W \cong \mathrm{Hom}_{\mathbb{C}}(V, W)$ if $\dim V < \infty$.

(c) Does part (b) hold if $\dim V = \infty$?