Algebra Qualifying Exam, January 2022

Do 5 of the following problems, including at least one from each of parts A, B, and C.

Part A

1. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is a prime.
   (a) Show that G has order $(p^2 - 1)(p^2 - p)$.
   (b) Show that for $p = 2$ the group G is isomorphic to the symmetric group S_3.
2. Let p and q be distinct prime numbers.
   (a) Show that if G is a group of order p^2q, then G is not simple.
   (b) Is every group of order p^2q a semidirect product of proper subgroups? Prove your answer.
3. Let G be a group, and let H be a subgroup of G of index m.
   (a) Show that there exists a homomorphism $\phi : G \to S_m$ whose kernel is a subgroup of H.
   (b) If m is the smallest prime dividing the order of G, prove that H is normal by showing that it is the kernel of ϕ.

Part B

4. (a) Determine the primes p for which $\mathbb{F}_p[X]/(X^2 + 1) \cong \mathbb{F}_p \times \mathbb{F}_p$. Prove your answer.
   (b) If $\mathbb{F}_p[X]/(X^2 + 1) \ncong \mathbb{F}_p \times \mathbb{F}_p$, is this quotient necessarily a field? Justify your answer.
5. (a) If F is a field, show that $F[X]$ is a PID.
   (b) Is $\mathbb{Z}[X]$ a PID? Prove your answer.
6. Let R be a commutative ring with 1. Prove that the center of the matrix ring $M_n(R)$ is the set of scalar matrices $\{aI_n : a \in R\}$.

Part C

7. Let R be a ring. An R-module N is called *simple* if it is not the zero module and if it has no submodules except N and the zero submodule.
   (a) Prove that any simple module N is isomorphic to R/M, where M is a maximal left ideal.
   (b) Prove *Schur's Lemma:* Let $\varphi : S \to S'$ be a homomorphism of simple modules. Then either φ is zero, or it is an isomorphism.
8. Let R be a PID, and let M and N be two finitely generated R-modules. If $M \oplus M \cong N \oplus N$, show that $M \cong N$.