Algebra Comprehensive Examination
August 2002

Directions. Choose six out of the ten problems below. Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. Hand in ONLY the six problems you want graded. You have 2 and 1/2 hours for this test. Good luck!

(1) Let G be an abelian group and let $G_{\text{tors}} \subset G$ be the subset of elements of finite order.

(a) Show that G_{tors} is a subgroup (called the *torsion subgroup of* G).

(b) Show that if $f : G \to G'$ is a homomorphism of abelian groups, then $f(G_{\text{tors}}) \subset G'_{\text{tors}}$.

(c) Determine the torsion subgroup of the group $G = \mathbf{R}/\mathbf{Z}$ (the group law on $\mathbf{R}$ and $\mathbf{Z}$ is addition).

(2) A *Möbius transformation* is a rational function $f \in \mathbf{C}(z)$ of the form $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbf{C}$ satisfy $ad - bc = 1$. It is easy to check that composition of functions makes the set M of all Möbius transformations into a group (composition is defined by $(f \circ g)(z) = f(g(z))$ for any rational functions f, g).

(a) Let $\phi : SL_2(\mathbf{C}) \to M$ be the map given by $\phi(A)(z) = \dfrac{az+b}{cz+d}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{C})$. Show that ϕ is a group homomorphism.

(b) Show that ϕ induces a group isomorphism $\bar{\phi} : PSL_2(\mathbf{C}) \to M$. (Recall that $PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\{\pm I\}$.)

(3) Let $<,>$ denote the standard inner product on $\mathbf{R}^n$. Let $v \in \mathbf{R}^n$ be a vector with $< v, v >= 1$ and let $\sigma_v : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation given by $\sigma(x) = x - 2 < x, v > v$.

(a) Show that σ_v is an orthogonal operator and that it satisfies $\sigma_v^2 = 1_{\mathbf{R}^n}$.

(b) Show that there is an orthonormal basis of $\mathbf{R}^n$ such that the matrix of σ_v in this basis is of the form

$$\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

(4) Let R be a commutative ring. An element $x \in R$ is called *nilpotent* if there is an integer $n \geq 1$ such that $x^n = 0$. The subset $N \subset R$ of all nilpotent elements is called the *nilradical* of R.

(a) Show that the nilradical N is an ideal of R and that N is contained in all prime ideals of R.

(b) Determine the nilradical of the ring $\mathbf{Z}/72\mathbf{Z}$. Deduce a method to determine the nilradical of $\mathbf{Z}/n\mathbf{Z}$ for any n.

(5) The set $C(\mathbf{R})$ of all continuous functions $f : \mathbf{R} \to \mathbf{R}$ is a commutative ring with unity for the usual operations of addition and multiplication of functions. Let $I = \{f \in C(\mathbf{R}) : \text{there exists } \epsilon > 0 \text{ such that } f(x) = 0 \text{ if } |x| < \epsilon\}$.

(a) Show that I is an ideal of $C(\mathbf{R})$

(b) Show that the quotient ring $C(\mathbf{R})/I$ has a unique maximal ideal (a ring with this property is called *local*). *Hint:* Show that the ideal $M = \{f \in C(\mathbf{R}) : f(0) = 0\}$ of $C(\mathbf{R})$ is maximal and use the correspondence theorem.

(6) Let $F = \left\{ \begin{pmatrix} x & -3y \\ y & x \end{pmatrix} \ : \ x, y \in \mathbf{Q} \right\}$.

(a) Show that F is a field under the usual matrix operations of addition and multiplication.

(b) Show that F is isomorphic to $\mathbf{Q}(\sqrt{-3}) = \{x + y\sqrt{-3} : x, y \in \mathbf{Q}\} \subset \mathbf{C}$.

(7) (a) State the Structure Theorem for Finitely Generated Abelian Groups

(b) Determine whether the finite abelian groups $A = \mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$ and $B = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/24\mathbf{Z}$ are isomorphic. Justify your answer.

(8) Determine the structure of the abelian group (as in the Structure Theorem for Finitely Generated Abelian Groups) generated by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, subject to the relations

$$\begin{aligned} 8\mathbf{x} + 12\mathbf{y} + 4\mathbf{z} &= 0 \\ 14\mathbf{x} + 24\mathbf{y} + 8\mathbf{z} &= 0 \\ 4\mathbf{x} + 24\mathbf{y} + 4\mathbf{z} &= 0. \end{aligned}$$

(9) (a) Let F be a field and let $p, q \in F[X]$ be distinct irreducible monic polynomials. How many conjugacy classes of matrices A are there with characteristic polynomial p^2q^3?

(b) List representatives of all conjugacy classes of matrices in $M_5(F)$ with characteristic polynomial $f = X^2(X+1)^3$.

(10) Let $N : \mathbf{C}^n \to \mathbf{C}^n$ be a nilpotent operator. (Recall that N is *nilpotent* if $N^r = 0$ for some integer $r \in \mathbf{N}$).

(a) Show that the number of Jordan blocks in the canonical Jordan form of N is equal to $\dim \ker(N)$.

(b) Show that the size of the largest Jordan block of N is equal to the nilpotency index of N (Recall that the *nilpotency index* of N is the smallest $r \in \mathbf{N}$ such that $N^r = 0$).

(c) Find the Jordan canonical form of the nilpotent operator $N : \mathbf{C}^6 \to \mathbf{C}^6$ given by the matrix

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 4 & 2 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -2 & -1 \end{pmatrix}.$$