Comprehensive Examination
Core-1 Algebra
August 2003

Directions: *Do exactly one problem from the set {G1, G2} (on groups); exactly one problem from the set {R1, R2} (on rings); exactly one problem from the set {M1, M2} (on modules); and exactly one problem from the set {L1, L2} (on linear algebra). Turn in a total of* **four** *problems.* State precisely any theorems you quote. Even if you are unable to do part (a) of some problem, you may use part (a) when proving part (b) of that problem; likewise for parts (b), (c), (d), etc. *Partial credit is possible.* Here, $\mathbb{Z}$ and $\mathbb{Q}$ denote the rings of integers and rational numbers, respectively. Please start each problem on a new sheet of paper, with your name and the problem number written at the top of every sheet. You have two hours, plus 30 minutes "overtime," for a total of $\mathbf{2\frac{1}{2}}$ **hours**. Good Luck!

G1. (a) List all Abelian groups of order 400 (up to isomorphism). Brief justification.

(b) Give the elementary divisors and invariant factors of the group $Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_{25}$. (Here Z_n denotes the cyclic group of order n.)

G2. Let G be a group (written multiplicatively). Let n be a positive integer. Let $G_n = \{ g^n \mid g \in G \}$. Let $\overline{G_n}$ be the intersection of all subgroups of G containing G_n (i.e., the subgroup of G generated by G_n). Prove:

(a) The subgroup $\overline{G_n}$ of G is normal. (You may assume that $\overline{G_n}$ is a subgroup.)

(b) Every element of $G/\overline{G_n}$ has finite order.

(c) $G/\overline{G_2}$ is Abelian.

(d) Give an example of a group G and an integer $n \geq 1$ such that $G/\overline{G_n}$ is not Abelian.

R1. Let R be a principal ideal domain. Use the following definitions: A *divisor* (in R) of $a \in R$ is any element $d \in R$ such that $dx = a$ for some $x \in R$; a *common divisor* of a and $b \in R$ is any element of R that is a divisor of both a and b; a *greatest common divisor* (in R) of a and b is a common divisor of a and b that is divisible by every common divisor of a and b. Prove the following:

(a) For two elements $a, b \in R$, a greatest common divisor exists and can be expressed as $ax + by$, for some $x, y \in R$.

[*Comment*: We do not assume that there is a Euclidean function, so the Euclidean algorithm is not applicable here.]

(b) Suppose d_1 and d_2 are both greatest common divisors of a and $b \in R$. State and prove a relation that exists between d_1 and d_2.

(c) Suppose that $a \in R$ is not a unit, and that the only divisors of a are elements of the form ua with u a unit of R. Prove that $R/(a)$ is a field.

R2. (Chinese Remainder Theorem) Let R be a commutative ring with 1, and let I and J be ideals of R such that $R = I + J$. Prove that $IJ = I \cap J$, and that there is a ring isomorphism

$$R/IJ \cong (R/I) \times (R/J).$$

M1. Let R be a ring with 1. Prove that a unitary left R-module M is simple if and only if $M \cong R/I$, for some maximal left-ideal I. (Recall, a unitary left R-module M is called *simple* if it is nonzero and it has no submodules other than M and (0).) *Hint*: For the "only if" direction, begin by proving that every simple module is cyclic.

M2. Let $\mathbb{Z}[\frac{1}{2}]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Below we shall also view $\mathbb{Z}[\frac{1}{2}]$ as a $\mathbb{Z}$-module.

(a) Is $\mathbb{Z}[\frac{1}{2}]$ "finitely generated" as a subring of $\mathbb{Q}$? (I.e., is there a finite subset $S \subset \mathbb{Q}$ such that $\mathbb{Z}[\frac{1}{2}]$ is the smallest subring of $\mathbb{Q}$ containing S? Brief justification.)

(b) Is $\mathbb{Z}[\frac{1}{2}]$ finitely generated as a $\mathbb{Z}$-module? (Brief justification.)

(c) Is $\mathbb{Z}[\frac{1}{2}]$ free as a $\mathbb{Z}$-module? (Brief justification, including a definition of "free.")

L1. Let

$$A = \begin{bmatrix} -4 & -2 & -4 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix},$$

viewed as a matrix over $\mathbb{Q}$.

(a) Show that A is nilpotent.

(b) Find the minimal polynomial of A.

(c) Find the characteristic polynomial of A.

(d) Find the Jordan canonical form of A.

L2. Let V be a vector space over a field F.

(a) Let $S \subset V$ be a linearly independent set. Show that there exists a basis of V containing S. (Hint: Zorn's lemma.)

(b) Show that any two finite bases of V over F have the same number of elements.