Directions: Do exactly one problem from the set \{G1, G2\} (on groups); exactly one problem from the set \{R1, R2\} (on rings); exactly one problem from the set \{M1, M2\} (on modules); and exactly one problem from the set \{L1, L2\} (on linear algebra). Turn in a total of four problems. State precisely any theorems you quote. Even if you are unable to do part (a) of some problem, you may use part (a) when proving part (b) of that problem; likewise for parts (b), (c), (d), etc. Partial credit is possible. Here, \(\mathbb{Z}\) and \(\mathbb{Q}\) denote the rings of integers and rational numbers, respectively. Please start each problem on a new sheet of paper, with your name and the problem number written at the top of every sheet. You have two hours, plus 30 minutes “overtime,” for a total of 2\(\frac{1}{2}\) hours. Good Luck!

G1. (a) List all Abelian groups of order 400 (up to isomorphism). Brief justification.

(b) Give the elementary divisors and invariant factors of the group \(\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25}\). (Here \(\mathbb{Z}_n\) denotes the cyclic group of order \(n\).)

G2. Let \(G\) be a group (written multiplicatively). Let \(n\) be a positive integer. Let \(G_n = \{g^n \mid g \in G\}\). Let \(\overline{G_n}\) be the intersection of all subgroups of \(G\) containing \(G_n\) (i.e., the subgroup of \(G\) generated by \(G_n\)). Prove:

(a) The subgroup \(\overline{G_n}\) of \(G\) is normal. (You may assume that \(\overline{G_n}\) is a subgroup.)

(b) Every element of \(G/\overline{G_n}\) has finite order.

(c) \(G/\overline{G_2}\) is Abelian.

(d) Give an example of a group \(G\) and an integer \(n \geq 1\) such that \(G/\overline{G_n}\) is not Abelian.

R1. Let \(R\) be a principal ideal domain. Use the following definitions: A divisor (in \(R\)) of \(a \in R\) is any element \(d \in R\) such that \(dx = a\) for some \(x \in R\); a common divisor of \(a\) and \(b \in R\) is any element of \(R\) that is a divisor of both \(a\) and \(b\); a greatest common divisor (in \(R\)) of \(a\) and \(b\) is a common divisor of \(a\) and \(b\) that is divisible by every common divisor of \(a\) and \(b\). Prove the following:

(a) For two elements \(a, b \in R\), a greatest common divisor exists and can be expressed as \(ax + by\), for some \(x, y \in R\).

[Comment: We do not assume that there is a Euclidean function, so the Euclidean algorithm is not applicable here.]

(b) Suppose \(d_1\) and \(d_2\) are both greatest common divisors of \(a\) and \(b \in R\). State and prove a relation that exists between \(d_1\) and \(d_2\).

(c) Suppose that \(a \in R\) is not a unit, and that the only divisors of \(a\) are elements of the form \(ua\) with \(u\) a unit of \(R\). Prove that \(R/(a)\) is a field.
R2. (Chinese Remainder Theorem) Let $R$ be a commutative ring with 1, and let $I$ and $J$ be ideals of $R$ such that $R = I + J$. Prove that $IJ = I \cap J$, and that there is a ring isomorphism

$$R/IJ \cong (R/I) \times (R/J).$$

M1. Let $R$ be a ring with 1. Prove that a unitary left $R$-module $M$ is simple if and only if $M \cong R/I$, for some maximal left-ideal $I$. (Recall, a unitary left $R$-module $M$ is called simple if it is nonzero and it has no submodules other than $M$ and $(0)$.) **Hint:** For the “only if” direction, begin by proving that every simple module is cyclic.

M2. Let $\mathbb{Z}[\frac{1}{2}]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Below we shall also view $\mathbb{Z}[\frac{1}{2}]$ as a $\mathbb{Z}$-module.

(a) Is $\mathbb{Z}[\frac{1}{2}]$ “finitely generated” as a subring of $\mathbb{Q}$? (I.e., is there a finite subset $S \subset \mathbb{Q}$ such that $\mathbb{Z}[\frac{1}{2}]$ is the smallest subring of $\mathbb{Q}$ containing $S$? Brief justification.)

(b) Is $\mathbb{Z}[\frac{1}{2}]$ finitely generated as a $\mathbb{Z}$-module? (Brief justification.)

(c) Is $\mathbb{Z}[\frac{1}{2}]$ free as a $\mathbb{Z}$-module? (Brief justification, including a definition of “free.”)

L1. Let

$$A = \begin{bmatrix} -4 & -2 & -4 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix},$$

viewed as a matrix over $\mathbb{Q}$.

(a) Show that $A$ is nilpotent.

(b) Find the minimal polynomial of $A$.

(c) Find the characteristic polynomial of $A$.

(d) Find the Jordan canonical form of $A$.

L2. Let $V$ be a vector space over a field $F$.

(a) Let $S \subset V$ be a linearly independent set. Show that there exists a basis of $V$ containing $S$. (Hint: Zorn’s lemma.)

(b) Show that any two finite bases of $V$ over $F$ have the same number of elements.