Core I Algebra   August 2006

Directions. The first three starred problems are required. Then choose two of the next five problems, for a total of five. Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet you turn in. Turn in only the five problems that you want graded. You have two and a half hours for the exam. Good luck!

1*. Give an example of each of the following. No proofs are required for this question.

(a) A nonabelian group of order 12.

(b) An R-module M which is not free.

(c) A prime ideal P in a ring R which is not principal.

(d) Two subgroups H and K of a group G, one of them normal, one of them not normal.

(e) An endomorphism T of a vector space V over a field k that cannot be diagonalized.

2*. Answer these questions:

(a) Up to isomorphism, list all the abelian groups of order 108.

(b) Write an explicit isomorphism between the multiplicative group $(\mathbf{Z}/12)^*$ of integers mod 12 relatively prime to 12 and the additive group $\mathbf{Z}/2 \times \mathbf{Z}/2$. Justify.

3*. Let

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ -1 & -3 & -1 \end{pmatrix}$$

(a) Compute the characteristic polynomial $c_A(x)$ and the minimal polynomial $m_A(x)$ of A.

(b) Determine the rational canonical form of the matrix A over the field $\mathbf{Q}$ of rational numbers.

(d) Determine the eigenvalues of the matrix A and their corresponding eigenspaces.

(d) Determine the Jordan canonical form of the matrix A.

4. Consider the alternating group S_3. Let

$$K = \{e = \text{identity}, \lambda = (123), \mu = (132)\}$$

Show that K is a normal subgroup of S_3, and determine the quotient S_3/K. Verify: K is both a subgroup and it is normal.

5. Let G be a group with 55 elements. Show that G has a normal subgroup of order 11. Hint: Sylow theorems.

6. Determine all the similarity classes of 4 by 4 matrices with entries from the field $\mathbf{Q}$, whose minimal polynomial is $(x-1)^2(x-2)$

7. Let F be a field and $R = F[x]$. Let $0 \neq f(x) \in R$. Show that the following are equivalent:

i. $f(x)$ is irreducible.

ii. The ideal $(f(x))$ is prime.

ii. The ideal $(f(x))$ is maximal.

8. Let $\mathbf{F}_3 = \mathbf{Z}/3$. Show that x^2+1 is irreducible in $\mathbf{F}_3[x]$. Explain why

$$K = \mathbf{F}_3[x]/(x^2+1)$$

is a field. How many elements does it have? In this field, write the inverse $1/\alpha$ in the form $a + b\alpha$ with $a, b, \in \mathbf{F}_3$. Here, α is the class of x in the quotient K.