**Directions.** Do exercises 1, 2, and 3, plus any two of the remaining four problems. *Start each problem on a new sheet of paper, and put your name and the problem number at the top of every sheet. Hand in **only** the five problems that you want graded.* The time available for the exam is two and one-half hours. Good luck!

1. Let $H$ be a normal subgroup of a group $G$, and let $K$ be a subgroup of $H$.

   (a) Give an example of this situation where $K$ is *not* a normal subgroup of $G$.

   (b) Prove that if the normal subgroup $H$ is cyclic, then $K$ is a normal subgroup of $G$.

2. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$. Answer the following questions (with proof) about $R$.

   (a) Why is $R$ an integral domain?

   (b) What are the units in $R$?

   (c) Is the element 2 irreducible in $R$?

   (d) Is 2 a prime element of $R$?

3. Let

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Find the characteristic polynomial $c_A(X)$ and minimal polynomial $m_A(X)$.

   (b) For each eigenvalue $\lambda$ of $A$, find the algebraic multiplicity $\nu_{\text{alg}}(\lambda)$ and the geometric multiplicity $\nu_{\text{geom}}(\lambda)$.

   (c) Find the Jordan canonical form $J$ of the matrix $A$.

   (d) Find an invertible matrix $P$ such that $P^{-1}AP = J$.

4. Let $R$ be an integral domain and let $M$ be an $R$-module. Give the definition of each of the following terms:

   (a) $M$ is a *free* $R$-module.

   (b) $M$ is a *cyclic* $R$-module.

   Now determine if each of the following statements about $R$-modules is true or false. Give a proof if true or, if false, give a counterexample and be sure to prove that your counterexample is a counterexample.

   (c) A submodule of a free module is free.

   (d) A submodule of a cyclic module is cyclic.

   (e) A quotient module of a free module is free.

   (f) A quotient module of a cyclic module is cyclic.

5. Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces $V$ and $W$. Show that $\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = \dim V$.

6. Let $R$ be a commutative ring with identity and let $I$ and $J$ be ideals of $R$. Define a subset of $R$ by
$$(I : J) = \{r \in R : rx \in I \text{ for all } x \in J\}.$$
   (a) Show that $(I : J)$ is an ideal of $R$ containing $I$.
   (b) Show that if $P$ is a prime ideal of $R$ and $x \notin P$, then $(P : \langle x \rangle) = P$, where $\langle x \rangle$ denotes the principal ideal generated by $x$.

7. List without repetition all of the abelian groups of order $72 = 3^2 2^3$, in elementary divisor form. Identify which group on your list is isomorphic to each of the following groups.
   (a) $\mathbb{Z}_{72}$
   (b) $\mathbb{Z}_4 \times \mathbb{Z}_{18}$
   (c) $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6$