Mathematics Comprehensive Examination
Core-1 ALGEBRA
August 2004

DIRECTIONS: Do the first three problems and choose two problems from the last three problems, for a total of FIVE out of the six problems below.

Start each problem on a new sheet of paper. Please put your name and the problem number at the top of every sheet. Hand in ONLY five problems.

You have two and a half hours. GOOD LUCK!

1. Find the number of nonisomorphic abelian groups of order $2^5 \cdot 3^4 \cdot 5^3$ that have an element of order eight.

2. a. Produce the lattice of subgroups of the alternating group A_4.

   b. Find five subgroups of order four of the alternating group A_5.

3. Let $\mathbb{F}_3$ be the field with three elements. Show that $f = X^3 + 2X + 2$ is irreducible in $\mathbb{F}_3[X]$. Deduce that $\mathbb{F}_3[X]/(f)$ is a field of order 27.

   The multiplicative group of the field $\mathbb{F}_3[X]/(f)$ is cyclic of order 26. Find a generator of that cyclic group.

4. Let $F = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \epsilon \mathbb{R} \right\}$.

   Show that F is a field under the usual matrix operations of addition and multiplication.

   Is F isomorphic to the field $\mathbb{C} = \{x + yi : x, y \epsilon \mathbb{R}\}$ of complex numbers? Justify your answer.

5. State the structure theorem for finitely generated modules over a PID and its special case concerning finitely generated abelian groups.

   Deduce that every finitely generated torsion-free module over a PID is free of finite rank, and every finitely generated torsion-free abelian group is isomorphic to $\mathbb{Z}^n$ for some n.

6. Let $M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & m & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   Determine all elements m of the field $\mathbb{Q}$ of rational numbers, for which the matrix M is diagonalizable over $\mathbb{Q}$.