Directions. Do the first exercise and any four of the remaining six problems. Start each problem on a new sheet of paper, and put your name and the problem number at the top of every sheet. Hand in *only* the five problem that you want graded. The time available for the exam is two and one-half hours. Good luck!

1. Give an example of each of the following. No proofs are required *for this exercise only.*
   (a) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose minimal polynomial is $m_T(X) = X^2 - 2X + 1$.
   (b) An element σ of order 12 in the alternating group A_{10}.
   (c) Two nonisomorphic groups of order 30.
   (d) A commutative ring R and an ideal I of R that is prime but not maximal.
   (e) A unique factorization domain (UFD) that is not a principal ideal domain (PID).

2. Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. Let

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : ad \neq 0 \right\} \subseteq G$$

   be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbb{R}$ is arbitrary.
   (a) Show that U is a normal subgroup of G and that G/U is isomorphic to D.
   (b) True or False (with justification): $G \cong U \times D$

3. Let $\mathbb{F}$ be a field and let R be the following subring of the ring of 2×2 matrices $M_2(\mathbb{F})$:

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,\, b,\, c, \in \mathbb{F} \right\},$$

   and let $J = \left\{ \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} : d \in \mathbb{F} \right\} \subseteq R$. Show that J is a (two-sided) ideal of R and that there is a ring isomorphism

$$\varphi : R/J \to \mathbb{F} \times \mathbb{F},$$

   where, as usual, $\mathbb{F} \times \mathbb{F}$ denotes the cartesian product of $\mathbb{F}$ with itself (as a ring).

4. Let $\mathbb{F}$ be a field, V a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear transformation whose minimal polynomial $m_T(X) \in \mathbb{F}[X]$ is *irreducible.* Use T to make the vector space V into an $\mathbb{F}[X]$-module, denoted V_T, in the usual way by defining the scalar multiplication $f(X) \cdot v = f(T)(v)$ for all $f(X) \in \mathbb{F}[X]$ and $v \in V$.
   (a) Let $v \in V$ be a nonzero vector and let $V_1 = \mathbb{F}[X]v$ be the cyclic submodule of V_T generated by v. Show that V_1 is the subspace of V spanned by v and the vectors $T^n(v)$ for all positive $n \in \mathbb{Z}$, and prove that $\dim_{\mathbb{F}} V_1 = \deg(m_T(X))$.
   (b) Prove that $\deg(m_T(X))$ divides $\dim_{\mathbb{F}} V$.

5. Let H_1 be the subgroup of $\mathbb{Z}^2$ generated by (6, 6) and (6, 4) and let H_2 be the subgroup of $\mathbb{Z}^2$ generated by (3, 1) and (3, 5). Determine if the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ are isomorphic.

6. Consider the following three matrices in $M_3(\mathbb{C})$.

$$\text{(i)} \begin{bmatrix} 3 & 0 & 0 \\ 3 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Find the characteristic polynomial $c_A(X)$ for each of the matrices above.

   (b) Find the minimal polynomial $m_A(X)$ for each of the matrices above.

   (c) Find the Jordan canonical form for each of the matrices above.

7. Let R be a commutative ring with identity, and let I and J be ideals of R.

   (a) Define what is meant by the *sum* $I + J$ and the *product* IJ of the ideals I and J.

   (b) Define *maximal ideal.*

   (c) If I and J are distinct maximal ideals, show that $I + J = R$ and $I \cap J = IJ$.