Core I Algebra   January 2006

Directions. The first three starred problems are required. Then choose two of the next five problems, for a total of five. Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet you turn in. Turn in only the five problems that you want graded. You have two and a half hours for the exam. Good luck!

1*. Give an example of each of the following. No proofs are required for this question.

(a) An ideal I in a ring R which is not principal.

(b) An R-module M which is not free.

(c) A prime ideal P in a ring R which is not maximal.

(d) Two nonisomorphic noncommutative groups of order 8.

(e) Two elements σ and τ in the symmetric group S_5 that are of the same order but not conjugate.

2*. Answer these questions:

(a) Up to isomorphism, list all the abelian groups of order 100.

(b) Is $\mathbf{Z}/36$ isomorphic with $\mathbf{Z}/4 \times \mathbf{Z}/9$? Justify.

(c) Is $\mathbf{Z}/64$ isomorphic with $\mathbf{Z}/16 \times \mathbf{Z}/4$? Justify.

3*. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 6 & -6 & 5 \end{pmatrix}$$

(a) Compute the characteristic polynomial $c_A(x)$ and the minimal polynomial $m_A(x)$ of A.

(b) Determine the rational canonical form of the matrix A over the field $\mathbf{Q}$ of rational numbers.

(d) Determine the eigenvalues of the matrix A and their corresponding eigenspaces.

(d) Determine the Jordan canonical form of the matrix A.

4. Consider the alternating group A_4. Let

$$V = \{e = \text{identity}, \lambda = (12)(34), \mu = (13)(24), \nu = (14)(23)\}$$

Show that V is a normal subgroup of A_4, and determine the quotient A_4/V. Verify: the given elements are in A_4 and V is both a subgroup and it is normal.

5. Let G be a group with 21 elements. Show that G has a normal subgroup of order 7. Hint: Sylow theorems.

6. Determine all the similarity classes of 5 by 5 matrices with entries from the field $\mathbf{Q}$, whose minimal polynomial is $(x-1)(x-2)^2$

7. Let F be a field and $R = F[x]$. Let $0 \neq f(x) \in R$. Show that the following are equivalent:

i. $f(x)$ is irreducible.

ii. The ideal $(f(x))$ is prime.

ii. The ideal $(f(x))$ is maximal.

8. Let $\mathbf{F}_5 = \mathbf{Z}/5$. Show that $x^2 + x + 1$ is irreducible in $\mathbf{F}_5[x]$. Explain why

$$K = \mathbf{F}_5[x]/(x^2 + x + 1)$$

is a field. How many elements does it have? In this field, write the inverse $1/\alpha$ in the form $a + b\alpha$ with $a, b, \in \mathbf{F}_5$. Here, α is the class of x in the quotient K.