Directions. Do exercises 1, 2, and 3, plus any two of the remaining four problems. *Start each problem on a new sheet of paper, and put your name and the problem number at the top of every sheet. Hand in **only** the five problems that you want graded.* The time available for the exam is two and one-half hours. Good luck!

1. Let H_1 be the subgroup of $\mathbb{Z}^2$ generated by $\{(1, 3), (1, 7)\}$ and let H_2 be the subgroup of $\mathbb{Z}^2$ generated by $\{(2, 4), (2, 6)\}$. Answer the following questions. Naturally, proofs are required.
   (a) Are the subgroups H_1 and H_2 isomorphic?
   (b) Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?

2. Let R be a commutative ring with identity, and let I and J be ideals of R.
   (a) Define what is meant by the *sum* $I + J$ and the *product* IJ of the ideals I and J.
   (b) Define *maximal ideal.*
   (c) If I and J are distinct maximal ideals, show that $I + J = R$ and $I \cap J = IJ$.

3. Let
$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & -1 & -2 \\ -1 & 0 & 0 \end{bmatrix}.$$
   (a) Find the characteristic polynomial $c_A(X)$ and minimal polynomial $m_A(X)$.
   (b) For each eigenvalue λ of A, find the algebraic multiplicity $\nu_{\text{alg}}(\lambda)$ and the geometric multiplicity $\nu_{\text{geom}}(\lambda)$.
   (c) Find the Jordan canonical form J of the matrix A.
   (d) Find an invertible matrix P such that $P^{-1}AP = J$.

4. Let H and N be subgroups of a group G with N normal. Prove that $HN = NH$ and that this set is a subgroup of G.

5. Let R be an integral domain. Recall that a nonzero element $p \in R$ that is not a unit is *prime* if $p|ab$ implies that $p|a$ or $p|b$, while p is *irreducible* if $p = ab$ implies that either a or b is a unit.
   (a) Prove that if p is prime then p is irreducible.
   (b) Prove that if R is a UFD (unique factorization domain) then every irreducible element is prime.

6. Let N be a submodule of an R-module M. Show that if N and M/N are finitely generated, then M is finitely generated.

7. Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be a linear transformation. Assume that the invariant factors of T are the two polynomials
$$f_1(X) = (X^2 - 1)(X - 3) \quad \text{and} \quad f_2(X) = (X^2 - 1)^2(X - 3)^3.$$
   (a) What is the minimal polynomial $m_T(X)$, characteristic polynomial $c_T(X)$, and, $\dim_{\mathbb{R}} V$, the dimension of V over $\mathbb{R}$?
   (b) Determine the Jordan canonical form for T.