Algebra	**Comprehensive/Qualifying Exam**	**January 2010**

Directions. Do five of the following six problems. Start each problem on a new sheet of paper, and put your name and the problem number at the top of every sheet. Hand in only the five problems that you want graded. You have 3 hours to complete this test. Good luck!

Problem 1. Let S_5 be the symmetric group on 5 elements.

(a) Find an element of S_5 of order 6.

(b) Show that there is no element of order 12 in S_5.

(c) Are the two elements

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

of S_5 conjugate? If yes, find an element giving the conjugation. If not, prove your answer (citing an appropriate theorem).

Problem 2. Let $R = \mathbb{Z}[X]$ be the polynomial ring in one variable with coefficients in the ring $\mathbb{Z}$ of integers. Let $I = (X^2 + 2, 7) \subset R$ be the ideal generated by $X^2 + 2$ and 7, and let $J = (X^2 + 1, 5) \subset R$ be the ideal generated by $X^2 + 1$ and 5. Answer the following questions and give a short proof of your answer in each case.

(a) Is I a maximal ideal? Is J a maximal ideal?

(b) Is I a prime ideal? Is J a prime ideal?

Problem 3.

(a) Show that every finite field has p^r elements where p is a prime and $r > 0$ an integer.

(b) Construct a field with 81 elements and show that your construction does indeed yield a field (with 81 elements).

Problem 4. Let $N \subset \mathbb{Z}^3$ be the submodule of $\mathbb{Z}^3$ generated by

$$\begin{pmatrix} 9 \\ 2 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 6 \\ 6 \\ 10 \end{pmatrix}.$$

Write the quotient $\mathbb{Z}$-module $\mathbb{Z}^3/N$ as a direct sum of cyclic $\mathbb{Z}$-modules. List elementary divisors and invariant factors of $\mathbb{Z}^3/N$.

Problem 5. Find, up to similarity, all matrices $A \in M_4(\mathbb{Q})$ satisfying

$$A^2 + A + I = 0$$

where $I \in M_4(\mathbb{Q})$ is the identity matrix. In each case, list the minimal polynomial, the characteristic polynomial, the rational canonical form and (if it exists) the Jordan canonical form of A.

Problem 6. Let G be a group of order $|G| = p^r$ where p is a prime and $r > 0$ an integer. Show that G has a non-trivial center, that is, $Z(G) \neq 1$.