Instructions: Do ***five (5)*** *problems, at least two from Part A, at least two from Part B, and one other from either part.* Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have 3 hours. *Good luck!*

Part A

1. Let G be a group and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian.

2. Let H_1 be the subgroup of $\mathbb{Z}^2$ generated by $\{(12, -8), (8, -4)\}$ and let H_2 be the subgroup of $\mathbb{Z}^2$ generated by $\{(-10, 4), (-4, 0)\}$. Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?

3. Let $\varphi : G \to H$ be a *surjective* group homomorphism and let N be a normal subgroup of G. Show that $\varphi(N)$ is a normal subgroup of H. What happens if φ is not surjective? Explain your answer.

4. Suppose a group G of order 2^n acts on a set X of odd cardinality. Show that there is some $x \in X$ whose stabilizer, $\{\, g \in G \mid gx = x \,\}$ is equal to G.

Part B

1. Let R be an integral domain containing a field k as a subring. Suppose that R is a finite-dimensional vector space over k, with scalar multiplication being the multiplication in R. Prove that R is a field.

2. Suppose that R is an integral domain and X is an indeterminate.
   (a) Prove that if R is a field, then the polynomial ring $R[X]$ is a PID (principal ideal domain).
   (b) Show, conversely, that if $R[X]$ is a PID, then R is a field.

3. Let R be a ring and M an R-module.
   (a) What does it mean for M to be a *free* R-module?
   (b) Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the **subring** of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}\left[\frac{1}{2}\right]$ is a free $\mathbb{Z}$-module.

4. Let R be an integral domain. Determine if each of the following statements about R-modules is true or false. Give a proof or counterexample, as appropriate.
   (a) A submodule of a free module is free.
   (b) A submodule of a free module is torsion-free.
   (c) A submodule of a cyclic module is cyclic.
   (d) A quotient module of a cyclic module is cyclic.
