Instructions: Complete any **five (5)** of the following problems. Turn in only these five problems to be graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

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1. Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. Let $D < G$ be the subgroup of invertible diagonal matrices and let $U < G$ be the subgroup of matrices of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, where $x \in \mathbb{R}$ is arbitrary.
   (a) Prove that U is a normal subgroup of G and that G/U is isomorphic to D.
   (b) True or False (with justification): $G \cong U \times D$.
2. Let S_{10} denote the symmetric group on 10 elements.
   (a) Find an element of order 21 in S_{10}.
   (b) What is the largest possible order of an element in S_{10}?
   (c) Prove that the alternating group A_{10} is normal in S_{10}. What is the quotient group S_{10}/A_{10}?
3. Let R be a ring, let $R^{\times}$ be the set of units of R, and let $M := R \setminus R^{\times}$. If M is an ideal, prove that is a maximal ideal, and that moreover it is the only maximal ideal of R.
4. (a) Find all ideals of the ring $\mathbb{Z}/24\mathbb{Z}$.
   (b) Find all ideals of the ring $\mathbb{Q}[x] \, / \left\langle x^2 + 2x - 2 \right\rangle$.
5. Suppose that M is an R-module, where R is a commutative ring with identity.
   (a) Prove that if M is a free R-module then the annihilator of M in R is 0.
   (b) Prove that M is a cyclic R-module if and only if $M \cong R/I$ for some ideal $I \subset R$.
6. (a) Find a basis and the invariant factors of the submodule N of $\mathbb{Z}^2$ generated by $x = (-6, 2), y = (2, -2)$ and $z = (10, 6)$.
   (b) From your answer to part (a), what is the structure of $\mathbb{Z}^2/N$?