Instructions: Complete any **five (5)** of the following problems. Turn in **only** these five problems to be graded. Write your name and the problem number at the top of each page that you turn in for grading. You have three hours. Good luck!

1. For $n \geq 1$, let S_n denote the symmetric group on n elements.
   (a) Prove that for any $n, m \geq 1$ there is a subgroup $H < S_{n+m}$ such that H is isomorphic to the direct product $S_n \times S_m$.
   (b) Prove that $S_4 \times S_3$ is **not** a subgroup of S_6.

2. Let G be a group and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian.
   (c) Let D_8 be the dihedral group of order 8, with center Z. Prove that D_8/Z is abelian but not cyclic.

3. (a) Prove that $\mathbb{R}[x]/\left(x^2+1\right)$ is isomorphic to the complex field.
   (b) Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, and define the quotient $R := \mathbb{Z}[i]/(3-4i)$. Determine which of the following rings R is isomorphic to, and justify your answer:

$$\mathbb{Z}/25\mathbb{Z} \qquad \text{or} \qquad \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}?$$

   (c) Is $(3-4i)$ a maximal ideal in $\mathbb{Z}[i]$?

4. Suppose that R is a commutative ring with identity.
   (a) Give the definition of a *unit* and a *nilpotent* element in R. Provide an example of each in the ring $\mathbb{Z}/100\mathbb{Z}$.
   (b) Prove that if $a \in R$ is a unit and $b \in R$ is nilpotent, then $a+b$ is a unit.

5. Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule of rank n. Prove that $\mathbb{Z}^n/M$ is a finite group.

6. (a) Let $A \in \mathrm{GL}_n(\mathbb{C})$ generate a cyclic multiplicative subgroup of finite order. Show that A is diagonalizable.
   (b) By considering $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, show that the result is false if $\mathbb{C}$ is replaced by $\mathbb{R}$.