Algebra **Comprehensive/Qualifying Exam** **January, 2017**

Instructions: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Determine the number of Sylow 3-subgroups of S_4.
   (b) Determine all the Sylow 3-subgroups of S_4.

2. Let H be a normal subgroup of a group G, and let K be a normal subgroup of H.
   (a) Give an example of this situation where K is not a normal subgroup of G.
   (b) Prove that if the normal subgroup H is cyclic, then K is normal in G.

Part B

3. Let $R = \mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} \mid a, b \in \mathbb{Z}\}$.
   (a) What are the units of R?
   (b) If $0 < p < 13$ is a prime integer, prove that p is irreducible in R.
   (c) Is 7 a prime element of R? Justify your answer.

4. Let $I = \langle X^2 + 1, 5\rangle$ and $J = \langle X^2 + 1, 7\rangle$ be ideals of $\mathbb{Z}[X]$.
   (a) Prove or disprove that I is a prime ideal.
   (b) Prove or disprove that J is a prime ideal.

5. Let R be a ring and M an R-module. Suppose $f : M \to M$ is an R-module endomorphism which is idempotent, that is, $f = f \circ f$. Prove that $M \cong \ker(f) \oplus \mathrm{Im}(f)$.

Part C

6. Let N be a $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ generated by $x = (10, 4, 16)$, $y = (-4, 4, 4)$ and $z = (8, 4, 16)$.
   (a) Find the invariant factors of N.
   (b) Determine the elementary divisor decomposition of $G = \mathbb{Z}^3/N$ up to isomorphism.

7. Let $A \in M_6(\mathbb{Q})$ with the minimal polynomial $m_A(X) = (X^2 + 1)(X - 1)^2$.
   (a) Determine all the possible rational canonical forms similar to A and their corresponding characteristic polynomials.
   (b) Determine all the possible Jordan canonical forms similar to A in $M_6(\mathbb{C})$.
   (c) If $\mathrm{rank}(A - I) = 3$, write down the Jordan canonical form similar to A in $M_6(\mathbb{C})$.
