**Instructions**: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

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## Part A

1. (a) Show that the alternating group $A_n$ is normal in the symmetric group $S_n$.

   (b) Let $\alpha$ and $\beta$ be conjugate elements of the symmetric group $S_n$. Suppose that $\alpha$ fixes at least two symbols. Prove that $\alpha$ and $\beta$ are conjugate via an element $\gamma$ of the alternating group $A_n$.

2. Classify all groups of order 225.

## Part B

3. (a) Show that the ring $\mathbb{Q}[X]/(X^2+2)$ is isomorphic to $\mathbb{Q}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbb{Q}\}$.

   (b) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

4. (a) How many monic degree-2 irreducible polynomials are there in $\mathbb{Z}/3\mathbb{Z}[x]$?

   (b) How many monic degree-3 irreducible polynomials are there in $\mathbb{Z}/3\mathbb{Z}[x]$?

5. Let $R$ be a commutative ring with identity. If $I \subset R$ is an ideal, then the radical of $I$, denoted $\sqrt{I}$, is defined by $\sqrt{I} := \{a \in R : a^n \in I \text{ for some positive integer } n\}$.

   (a) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

   (b) If $P$ is a prime ideal of $R$ and $r \in \mathbb{N}$, find $\sqrt{P^r}$ and justify your answer.

   (c) Find $\sqrt{I}$, where $I$ is the ideal $\langle 108 \rangle$ in the ring $\mathbb{Z}$ of integers.

## Part C

6. Let $R$ be a ring and let $f : M \to N$ be a surjective homomorphism of $R$-modules, where $N$ is a free $R$-module. Show that there exists an $R$-module homomorphism $g : N \to M$ such that $f \circ g = 1_N$. Show that $M = \mathrm{Ker}(f) \oplus \mathrm{Im}(g)$.

7. Consider the $\mathbb{Q}[X]$-module $V$ with $V$ being the column vector space $\mathbb{Q}^3$ and the action of $X$ given by $X \cdot v := A \cdot v$ for $v \in V$, where

$$A = \begin{bmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{bmatrix} \in M_3(\mathbb{Q}).$$

   (a) Find the invariant factors of the $\mathbb{Q}[X]$-module $V$ and its invariant factor decomposition up to isomorphism.

   (b) Consider $A \in M_3(\mathbb{C})$ and find the Jordan canonical form of $A$.

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