Algebra **Comprehensive/Qualifying Exam** **January, 2019**

Instructions:

1. Do 5 of the following problems, including at least one from Part A, one from Part B, and one from Part C.
2. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. Clip your papers together in numerical order of the problems chosen when finished.

You have three hours. Good luck!

Part A

1. (a) Let G be a group and $a,\ b \in G$. Suppose that the order of a is m and the order of b is n. If $ab = ba$ and $\gcd(m, n) = 1$, prove that the order of ab is mn.

   (b) How many solutions of $x^{30} \equiv 1 \mod 6200$ are there? ($6200 = 2^3 \cdot 5^2 \cdot 31$) Please justify your answer.

2. Let q be an odd prime and k be a positive integer.

   (a) Determine the number of Sylow p-subgroups of the Dihedral group $D_{2^k q}$ of order $2^k q$, for $p = 2$ and $p = q$. Please justify your answer.

   (b) Prove or disprove each Sylow 2-subgroup of $D_{2^k q}$ is a Dihedral group.

Part B

3. Let R be a PID. Prove or disprove the following statements.

   (a) A element p of R is a prime if and only if p is an irreducible.

   (b) An ideal in R is prime if and only if I is maximal.

4. (a) Show that the ring $\mathbb{Q}[x]/ < x^2 + 1 >$ is isomorphic to $\mathbb{Q}[i] := \{a + bi : a, b \in \mathbb{Q}\}$.

   (b) Denote $\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$. Is $\mathbb{Z}[i]/\langle 2 + i\rangle$ a field? Please justify your answer.

5. Let $\mathbb{F}$ be the finite field $\mathbb{Z}/3\mathbb{Z}$. Let $f(x) = x^2 + 1$ and $g(x) = x^2 + x + 2 \in \mathbb{F}[x]$.

   (a) Give an explicit field isomorphism $\mathbb{F}[x]/\langle f(x)\rangle \cong \mathbb{F}[x]/\langle g(x)\rangle$.

   (b) Find the order and the inverse of $x + 2 + \langle f(x)\rangle$ in the multiplicaitve group of $\mathbb{F}[x]/\langle f(x)\rangle$. Please justify your answer.

Part C

6. Let R be a ring. An R-module N is called *simple* if it is not the zero module and if it has no submodules except N and the zero submodule.

   (a) Prove that any simple module N is isomorphic to R/M, where M is a maximal ideal.

   (b) Prove *Schur's Lemma*: Let $\phi : S \longrightarrow S'$ be a homomorphism of simple modules. Then either ϕ is zero, or it is an isomorphism.

7. Let $\mathbb{F}$ be a field of characteristic not 2 and $V := M_2(\mathbb{F})$ the vector space of 2×2 matrices with entries in $\mathbb{F}$. Make V into $\mathbb{F}[x]$-module via linear operator

$$T : V \longrightarrow V \quad T(A) = A + A^t$$

   by defining $x \cdot v = T(v)$, where A^t is the transpose of A. Denote the resulting $\mathbb{F}[x]$-module by V_T.

   (a) Determine the annihilator of V_T, $\text{ann}(V_T)$.

   (b) Find the rational canonical form of T and determine Jordan canonical form of T if it exists.