Instructions: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Determine the number of Sylow p-subgroups of S_5 for each prime factor p of $|S_5|$. Please prove your assertion.

   (b) Identify the isomorphism class of each Sylow p-subgroup of S_5. Please prove your assertion.

2. (a) Let G be a nonabelian group and $Z(G)$ its center. Prove that $G/Z(G)$ cannot be a cyclic group.

   (b) Let p be a prime and G a nonabelian group of order p^3. Determine the isomorphism classes of $Z(G)$ and $G/Z(G)$. Prove your assertion.

Part B

3. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

   (a) Determine the units of R. Please prove your assertion.

   (b) Determine whether $3 + 2\sqrt{-5}$ is a prime element of R. Prove your assertion.

   (c) Identity the isomorphism class of the quotient ring $R/\langle 3+2\sqrt{-5}\rangle$. Please prove your assertion.

4. Let R be a commutative ring with identity, and I, J ideals of R.

   (a) If $I + J = R$, prove that $IJ = I \cap J$.

   (b) If R is a PID and $IJ = I \cap J$, prove that $I + J = R$.

   (c) Show that $\mathbb{Z}/810\mathbb{Z}$ is isomorphic to $\mathbb{Z}/81\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ as rings.

5. (a) Find all the ideals of the ring $\mathbb{Q}[X]/\langle X^4 - 2X^2 + 2\rangle$. Prove your assertion.

   (b) Find all the ideals of the ring $\mathbb{F}_5[X]/\langle X^4 - 2X^2 + 2\rangle$. Prove your assertion.

Part C

6. Let G be the abelian group with generators x, y, and z subject to the relations

$$\begin{array}{rcrcrcl} 4x & + & 20y & + & 4z & = & 0\,, \\ 8x & + & 16y & + & 2z & = & 0\,, \\ 2x & + & 4y & + & 2z & = & 0\,. \end{array}$$

Determine the invariant factors and elementary divisors of G and write G as a direct sum of cyclic groups. Please show the work for your assertions.

7. Let $f(X) = (X^2 - 2)(X - 1)^2 \in \mathbb{Q}[X]$.

   (a) Construct all the matrices A in $M_6(\mathbb{Q})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$ and determine their corresponding characteristic polynomials $\chi_A(X)$.

   (b) Construct all the Jordan canonical forms A in $M_6(\mathbb{C})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$.