Instructions: Do five (5) problems, including at least two (2) from Part A and at least two (2) from Part B. Start each chosen problem on a fresh sheet of paper. Write your name on each sheet at the top. Describe which theorems you apply, and check that all hypotheses are satisfied. Please turn in 5 problems, even if the solutions are imperfect, for partial credit. If you see an error or gap in your proof but are unsure how to fix at this time, then write this because it is better that you recognized it! Clip your papers together in numerical order of the problems chosen when finished. You have 3 hours. Good luck!

Symbols: Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is $l$, and $(X, \mathcal{A}, \mu)$ is an abstract measure space. A field or $\sigma$-field, $\mathcal{A}$, is also called an algebra or $\sigma$-algebra of sets, respectively. Integration is in the sense of Lebesgue. The $c$-translate of the set $S$ is $S + c = \{s + c \mid s \in S\}$. Disjoint unions are denoted by $\bigcup$.

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### Part A: Measures

1. Let $\mu$ be a finitely additive measure on a field $\mathfrak{A} \subseteq \mathcal{P}(X)$ such that $\mu(X) < \infty$. Prove that the following are equivalent:

   (a) If $A_n \in \mathfrak{A}$ for each $n$, and if $A = \bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A}$, then $\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

   (b) If $A_n \in \mathfrak{A}$ is such that $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then we have $\lim_{n \to \infty} \mu(A_n) = 0$.

2. (a) State Egoroff’s theorem for any measure space $(X, \mathfrak{A}, \mu)$ with $\mu(X) < \infty$.

   (b) Give an example of a sequence of Lebesgue measurable functions $f_n : [0, 1] \to \mathbb{R}$ that converges pointwise everywhere to a function $f$, having the property that there is no Lebesgue null-set $A$ of $[0, 1]$ for which $f_n$ converges uniformly to $f$ on the complementary set $[0, 1] \setminus A$. Explain why your example is as claimed, yet does not contradict part (a).

3. Let $p(x_1, x_2)$ be a polynomial in two variables, but not the zero polynomial. Prove that the set of points $x \in \mathbb{R}^2$ with $p(x) = 0$ has 2-dimensional Lebesgue measure $(l \times l)(p^{-1}(0)) = 0$.

   (Suggestion: Use Fubini’s theorem to evaluate $(l \times l)(p^{-1}(0))$.)

4. Prove that Lebesgue measure $l$ is translation-invariant on the real line. That is, prove, for each real number $c$, that

   (a) If $S$ is Lebesgue measurable, then so is $S + c$ and, moreover,

   (b) $l(S + c) = l(S)$. (Do not base your proof on the fact that the Lebesgue integral is translation-invariant, since that is usually established based upon the claim of the present exercise.)
Part B: Integrals

5. Let $f \in L^1[0, \infty)$. Prove carefully that

$$\lim_{n \to \infty} \int_0^n e^{-nx} f(x) \, dx = 0.$$ 

Be sure to explain and justify how you are using the limit theorem(s) you have chosen to apply.

6. Let $f \in \mathcal{L}(X, \mathfrak{A}, \mu)$ be a non-negative function on a measure space. Suppose that the sequence $\int_X f^n \, dl$ converges to a real number. Prove that the sequence of powers $f^n$ converges almost everywhere to the indicator function of a measurable set.

7. Let $f \in L^1[a, b]$ with respect to Lebesgue measure. Define $F(x) = \int_a^x f \, dl$ for each $x \in [a, b]$. Prove that $F$ is both absolutely continuous and of bounded variation on $[a, b]$.

8. Let $f \in L^1(\mathbb{R})$ and suppose also that $\int_{\mathbb{R}} |xf(x)| \, dl(x) < \infty$. Define

$$F(\alpha) = \int_{\mathbb{R}} f(x) \sin(\alpha x) \, dl(x)$$

for all $\alpha \in \mathbb{R}$. Prove that the derivative

$$F'(\alpha) = \frac{d}{d\alpha} \int_{\mathbb{R}} f(x) \sin(\alpha x) \, dl(x)$$

exists and find its value for all $\alpha \in \mathbb{R}$. (The point of this problem is to justify carefully bringing the differentiation with respect to $\alpha$ inside the integral.)