Instructions: Do ***five (5)*** *problems, including* ***at least two (2) from Part A*** and ***at least two (2)*** *from* ***Part B***. *Start each chosen problem on a fresh sheet of paper.* **Write your name on each sheet at the top**. Describe which theorems you apply, and check that all hypotheses are satisfied. **Please turn in 5 problems, even if the solutions are imperfect, for partial credit.** If you see an error or gap in your proof but are unsure how to fix at this time, then write this because it is better that you recognized it! *Clip your papers together in numerical order of the problems chosen when finished.* You have 3 hours. **Good luck!**
Symbols: Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is l, and $(X, \mathfrak{A}, \mu)$ is an abstract measure space. A *field* or σ-*field*, $\mathfrak{A}$, is also called an *algebra* or σ-*algebra* of sets, respectively. Integration is in the sense of Lebesgue. The c-translate of the set S is $S + c = \{s + c \mid s \in S\}$. *Disjoint* unions are denoted by $\dot{\bigcup}$.

Part A: Measures

1. Let μ be a *finitely additive* measure on a field $\mathfrak{A} \subseteq \mathfrak{P}(X)$ such that $\mu(X) < \infty$. Prove that the following are *equivalent*:
   (a) If $A_n \in \mathfrak{A}$ for each n, and if $A = \dot{\bigcup}_{n \in \mathbb{N}} A_n \in \mathfrak{A}$, then $\mu(A) = \sum_{n \in \mathbb{N}} \mu(A_n)$.
   (b) If $A_n \in \mathfrak{A}$ is such that $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then we have $\lim_{n \to \infty} \mu(A_n) = 0$.

2. (a) State Egoroff's theorem for any measure space $(X, \mathfrak{A}, \mu)$ with $\mu(X) < \infty$.
   (b) Give an example of a sequence of Lebesgue measurable functions $f_n : [0, 1] \to \mathbb{R}$ that converges pointwise everywhere to a function f, having the property that there is no Lebesgue null-set A of $[0, 1]$ for which f_n converges uniformly to f on the complementary set $[0, 1] \setminus A$. Explain why your example is as claimed, yet does not contradict part (a).

3. Let $p(x_1, x_2)$ be a polynomial in two variables, but not the zero polynomial. Prove that the set of points $x \in \mathbb{R}^2$ with $p(x) = 0$ has 2-dimensional Lebesgue measure $(l \times l)\left(p^{-1}(0)\right) = 0$. (Suggestion: Use Fubini's theorem to evaluate $(l \times l)\left(p^{-1}(0)\right)$.)

4. Prove that Lebesgue measure l is translation-invariant on the real line. That is, prove, for each real number c, that
   (a) If S is Lebesgue measurable, then so is $S + c$ and, moreover,
   (b) $l(S + c) = l(S)$. (Do not base your proof on the fact that the Lebesgue integral is translation-invariant, since that is usually established based upon the claim of the present exercise.)

Part B: Integrals

5. Let $f \in L^1[0,\infty)$. Prove carefully that

$$\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\, dx = 0.$$

Be sure to explain and justify how you are using the limit theorem(s) you have chosen to apply.

6. Let $f \in \mathcal{L}(X, \mathfrak{A}, \mu)$ be a non-negative function on a measure space. Suppose that the sequence $\int_{\mathbb{R}} f^n \, dl$ converges to a real number. Prove that the sequence of powers f^n converges almost everywhere to the indicator function of a measurable set.

7. Let $f \in L^1[a,b]$ with respect to Lebesgue measure. Define $F(x) = \int_a^x f\, dl$ for each $x \in [a,b]$. Prove that F is both absolutely continuous and of bounded variation on $[a,b]$.

8. Let $f \in L^1(\mathbb{R})$ and suppose also that $\int_{\mathbb{R}} |xf(x)|\, dl(x) < \infty$. Define

$$F(\alpha) = \int_{\mathbb{R}} f(x) \sin(\alpha x)\, dl(x)$$

for all $\alpha \in \mathbb{R}$. Prove that the *derivative*

$$F'(\alpha) = \frac{d}{d\alpha} \int_{\mathbb{R}} f(x) \sin(\alpha x)\, dl(x)$$

exists and find its value for all $\alpha \in \mathbb{R}$. (*The point of this problem is to justify carefully* bringing the differentiation with respect to α inside the integral.)