Instructions: Do ***five (5)*** *problems, including* ***at least two (2) from Part A*** and ***at least two (2)*** *from* ***Part B***. *Start each chosen problem on a fresh sheet of paper.* **Write your name on each sheet at the top**. *Describe which theorems you apply, and check that all hypotheses are satisfied.* **Please turn in 5 problems, even if the solutions are imperfect, for partial credit.** If you see an error or gap in your proof but are unsure how to fix it, then write this because it is better that you recognized it! When you are finished, *clip your papers together in numerical order of the problems chosen.* You have 3 hours. **Good luck!**

Symbols: Lebesgue measure on $\mathbb{R}^n$ is l, $n \geq 1$. If $S \subseteq \mathbb{R}^n$, then $S + c = \{s + c \mid s \in S\}$ and $-S = \{-s \mid s \in S\}$. Integration is in the sense of Lebesgue. $(X, \mathfrak{A}, \mu)$ is abstract measure space; $\dot\cup$ is *disjoint* union; $1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$; and $\nu \prec \mu$ means ν is absolutely continuous with respect to μ.

Part A: Measures

1. Let $(X, \mathfrak{A}, \mu)$ be a measure space for which $\mu(X) < \infty$.
   (a) Suppose f_n is a sequence of measurable functions such that $f_n \to f$ everywhere. Prove that $f_n \to f$ in measure, meaning that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies \mu\left\{x \,\middle|\, |f_n(x) - f(x)| > \epsilon\right\} < \epsilon$.
   (b) Give an example on $[0, 1]$ of a sequence of Lebesgue measurable functions f_n converging in measure, yet $f_n(x)$ diverges for all x. Justify that your example is correct.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with the property that

$$l(E \cap I) \leq \frac{l(I)}{2},$$

   for every open interval I. Prove that $l(E) = 0$. (Hint: If E is measurable, then the *outer* measure $l^*(E) = l(E)$.)

3. Let $f \geq 0$ be a *real-valued* Lebesgue measurable function on $\mathbb{R}^d$. Define $\mu(E) = \int_E f\, dl$.
   (a) Show that the measure $\mu \prec l$.
   (b) Show that μ is σ-finite. Note that this means there is a sequence E_n of measurable sets with $E = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n.

4. Suppose A and B are measurable subsets of $\mathbb{R}$, each one of strictly positive but finite measure. Prove that there exists a number $c \in \mathbb{R}$ such that $l((A+c) \cap B) > 0$. (Hints: Use the *outer measures* $l^*(A)$ and $l^*(B)$ in order to give a proof in terms of intervals. Alternatively, you could apply Fubini's theorem to the convolution $1_{-A} * 1_B(x) = \int_{\mathbb{R}} 1_{-A}(x - y) 1_B(y)\, dy$, justifying the use of Fubini's theorem.)

Part B: Integrals

5. Let $f \in L^1(X, \mathfrak{A}, \mu)$, a σ-finite measure space with $X = \dot{\bigcup}_{n\in\mathbb{N}} X_n$ such that $\mu(X_n) < \infty$ for each n. Let $A_n \in \mathfrak{A}$ be such that $\mu(A_n) \to 0$.

   (a) Prove: $\lim_{n\to\infty} \int_{A_n} f \, d\mu = 0$.

   (b) Prove or disprove: $\lim_{N\to\infty} \int_{\bigcup_{n\geq N} X_n} f \, d\mu = 0$.

6. Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$ and if

$$\sum_{n\in\mathbb{N}} \int_0^1 |f_n(x)| \, dx < \infty,$$

   then $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent almost everywhere to an integrable function f, and

$$\int_0^1 f(x) \, dx = \sum_{n\in\mathbb{N}} \int_0^1 f_n(x) dx.$$

7. Define $f(x) = \int_{\mathbb{R}} \cos(xy) \, g(y) \, dy$ for $x \in \mathbb{R}$ where g is an integrable function on $\mathbb{R}$. Show that f is continuous.

8. Suppose that f and g are in $L^1(\mathbb{R}^n)$ and let $h(x,y) = f(x-y)g(y)$. Show that $h \in L^1(\mathbb{R}^{2n})$. Use this to *show* that the function

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) \, dy$$

   is *defined* for almost all x and that $f * g \in L^1(\mathbb{R}^n)$. *Then show* that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

   (Hints: (i) You may assume in this problem that the function h is Lebesgue measurable on $\mathbb{R}^{2n}$. (ii) You may use Fubini's theorem and/or Tonelli's theorem, being very careful to show that all hypotheses are satisfied.)