Instructions: All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration are in the sense of Lebesgue on the real line. You have three hours. We wish you well!

Part I: Choose 2 of the following 3 problems

1. Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathfrak{A}, \mu)$ is a measure space. Prove that the set $S = \left\{x \,\middle|\, \lim_{n\to\infty} f_n(x) \text{ exists}\right\}$ is a measurable set.

2. Let E be a measurable set in $[0,1]$ and let $c > 0$. If $l(E \cap I) \geq cl(I)$, for all open intervals $I \subset [0,1]$ show that $l(E) = 1$. (Here l denotes Lebesgue measure on the real line.)

3. Let μ be a finite and finitely additive measure of a field $\mathfrak{A} \subset \mathfrak{P}(X)$. Prove that μ is countable additive on $\mathfrak{A}$ if and only if for each decreasing sequence $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ we have $\lim_{n\to\infty} \mu(A_n) = 0$.

Part II: Choose 2 of the following 3 problems

1. Let $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure. Suppose that $\int_{\mathbb{R}} |x|\,|f(x)|\,dx < \infty$. Show that the function $g(y) = \int_{\mathbb{R}} e^{ixy} f(x)\,dx$ is differentiable at every $y \in \mathbb{R}$.

2. Let $\chi_{[-n,n]}$ denote the characteristic function of the interval $[-n, n]$, $n \in \mathbb{N}$. Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x) \sin\left(\frac{\pi x}{n}\right)$, $x \in \mathbb{R}$.

   (a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$? Why or why not?

   (b) Show that
   $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$
   Are the assumptions of Lebesgue's dominated convergence theorem satisfied? Why or why not?

3. Let $f(x)$ be a real-valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Show that
   $$\lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x))\,d\mu = \mu\left\{x | f(x) \in \mathbb{Z}\right\}.$$

Part III: Choose 1 of the following 2 problems

1. Let f be a nonnegative integrable function on $[0,1]$ and let $I = \int_0^1 f(x)\,dx$. Show that

$$\sqrt{1+I^2} \leq \int_0^1 \sqrt{1+f^2(x)}\,dx \leq 1+I.$$

2. Show that the product of two absolutely continuous function on a closed finite interval $[a,b]$ is absolutely continuous.