Instructions: All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration are in the sense of Lebesgue on the real line. You have three hours. We wish you well!

Part I: Choose 2 of the following 3 problems

1. Show there are no countably infinite σ-algebras.

2. Let f_n be a sequence of continuous functions on $[0,1]$ that converges pointwise for all $x \in [0,1]$. Given $\epsilon > 0$ show there is a subset $A \subset [0,1]$, $m(A) < \epsilon$, and a positive number M such that
$$|f_n(x)| \leq M,$$
for all $x \in [0,1] \setminus A$.

3. Suppose A and B are measurable subsets of $\mathbb{R}^n$, each one of strictly positive but finite measure. Let ℓ denote Lebesgue measure on $\mathbb{R}^n$. Prove that there exists a vector $c \in \mathbb{R}^n$ such that
$$\ell((A+c) \cap B) > 0.$$
(Hints: Consider the outer measure of A and B. Or, alternatively, consider the convolution $1_{-A} * 1_B$.)

Part II: Choose 2 of the following 3 problems

4. Let f be a differentiable function on $[-1,1]$. Prove that
$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| \leq 1} \frac{1}{x} f(x)\, dx$$
exists.

5. Compute the following limit and justify your calculations:
$$\lim_{n \to \infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx.$$

6. Provide an example of a sequence $\{f_n\}$ of measurable functions on $[0,1]$ such that $f_n \to f$ almost everywhere, and $f_n \geq 0$, yet
$$\liminf \int_0^1 f_n\, d\mu \neq \int_0^1 f\, d\mu.$$

Part III: Choose 1 of the following 2 problems

7. Let $(X, \mathfrak{A}, \mu)$ be a measure space for which $\mu(X) < \infty$. Show

$$L^q(X, \mathfrak{A}, \mu) \subset L^p(X, \mathfrak{A}, \mu),$$

whenever $1 \leq p \leq q \leq \infty$.

8. Suppose $(X, \mathfrak{A}, \mu)$ is a measure space and $m(X) = 1$. Suppose f and g are positive measurable functions on X such that $fg \geq 1$. Show that

$$\int_X f \, d\mu(x) \int_X g \, d\mu(x) \geq 1.$$