Solve one of the problems (1)–(2), two of the problems (3)–(6) and two of the problems (7)–(9). Only turn in the solution to at most \textbf{five} problems.

In the following \( \lambda \) denotes the Lebesgue measure on one of the spaces \( \mathbb{R}^n \) and \( \lambda_k \) denotes the Lebesgue measure on \( \mathbb{R}^k \). If not otherwise stated, the statement that \( \mu \) is a measure means that \( \mu \) is \( \sigma \)-additive. The notation \( \mathbb{R} \) stands for the extended real numbers. The indicator function of a set \( A \) is denoted by \( \chi_A \).

Turn in all your work even if you do not finish a problem. You might get partial credit. Make sure that you have written your name on all pages that you turn in.

1. Let \((X, \mathcal{A})\) be a measurable space and let \( \mu : \mathcal{A} \to [0, \infty] \) be a \textbf{finitely additive} measure.

(a) Show that if \( \mu(X) < \infty \) then \( \mu \) is countably additive if and only if for every decreasing sequence \( \{A_j\}_{j \in \mathbb{N}} \) in \( \mathcal{A} \) we have
\[
\mu \left( \bigcap_{j=1}^{\infty} A_j \right) = \lim_{j \to \infty} \mu(A_j). \tag{1}
\]

(b) Give an example of a measure space \((X, \mathcal{A}, \mu)\), \((\mu \ \sigma\text{-additive})\) with \( \mu(X) = \infty \), and a sequence of decreasing measurable sets \( \{A_j\}_{j \in \mathbb{N}} \) in \( \mathcal{A} \) such (1) fails.

2. Let \((X, \mathcal{A}, \mu)\) be a measure space such that \( \mu(X) > 0 \). Let \( f : X \to \mathbb{R} \) be measurable and suppose that \( f \) is finite \( \mu \) almost everywhere. Show that there exists \( Y \in \mathcal{A} \) such that \( \mu(Y) > 0 \) and \( f \) is bounded on \( Y \).

3. Let \((X, \mathcal{A}, \mu)\) be a finite measure space. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of measurable functions \( f_n : X \to \mathbb{R} \) that converges almost everywhere to a measurable function \( f : X \to \mathbb{R} \). Show that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) in measure.

4. Determine if the following limits exist. If the limit exists, find the limit and justify your answer:

(a) \( \lim_{n \to \infty} \int_{0}^{n} \left( 1 - \frac{x}{n} \right)^n \, dx. \)

(b) \( \lim_{n \to \infty} \int_{\mathbb{R}^+} \frac{\sin(x/n)}{1 + x^2} \, dx. \)

(c) \( \lim_{n \to \infty} \int_{\mathbb{R}} f(x) g_n(x) \, d\lambda(x) \) where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( g_n(x) = n \chi_{[-1/2,1/2]}(nx). \)
5. For \( n = 1, 2, \ldots \) define
\[
f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n. \end{cases}
\]
Find
\[
\lim_{n \to \infty} f_n = f \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) \, d\lambda(x).
\]
Explain why your result does not contradict the Lebesgue dominated convergence theorem.

6. Let \((X, \mathcal{A}, \mu)\) be a measure space. Assume that \(f_n \in L^1(X, \mu)\) for \(n \in \mathbb{N}\) and that
\[
\sum_{n=1}^{\infty} \int_X |f_n(x)| \, d\mu(x) < \infty.
\]
Show that
a) The series \(\sum_{n=1}^{\infty} f_n(x)\) converges almost everywhere to a measurable function \(f\) on \(X\).
b) \(f \in L^1(X, \mu)\) and
\[
\int_X f(x) \, d\mu(x) = \sum_{n=1}^{\infty} \int_X f_n(x) \, d\mu(x).
\]

7. Let \(f, g \in L^1(\mathbb{R}, \lambda_1)\).
   
   a) Show that \(H(x, y) = f(x)g(y-x)\) is integrable on \(\mathbb{R}^2\) with respect to the Lebesgue measure \(\lambda_2\).
   
   b) Show that \(x \mapsto H(x, y)\) is integrable with respect to \(\lambda_1\), that \(G(y) = \int_\mathbb{R} H(x, y) \, d\lambda_1(x)\) is integrable on \(\mathbb{R}\) with respect to \(\lambda_1\), and that
   \[
   \|G\|_1 \leq \|f\|_1 \|g\|_1.
   \]

8. For \(-\infty < a < b < \infty\) let \(I = [a, b]\) and let \(\mu\) be a signed measure on \(I\) (with respect to the Borel \(\sigma\)-algebra). Let \(f(x) = \mu([a, x])\). Then \(f\) is of bounded variation.

9. Let \(Q = [0,1]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (\forall j = 1, \ldots, n) 0 \leq x_j \leq 1\}\). Let \(f : Q \to \mathbb{R}\) be a continuous function. Show that the Lebesgue measure of the set \(\{(x, f(x)) \mid x \in Q\} \subset \mathbb{R}^{n+1}\) is zero.