Instructions: There are three parts. You must turn in 2 problems from Part I, 2 problems from Part II, and 1 problem from Part III. All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. *Unless otherwise indicated* all references to measure and integration on the real line are in the sense of Lebesgue. You have 3 hours to complete this exam. We wish you well!

Part I: Choose 2 of the following 3 problems

1. Let $f : X \to \mathbb{R}^*$ be an extended real-valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Suppose that $f(x)$ is finite for almost all x. Prove that for each $\epsilon > 0$ there exists a set $A \in \mathfrak{A}$, with $\mu(X \setminus A) < \epsilon$, such that f is bounded on A. Give an example for which it is impossible to require that $X \setminus A$ be a μ-null set.

2. Show there are no bounded sequences $\{a_n\}$ and $\{b_n\}$ for which
$$a_n \sin(2\pi nx) + b_n \cos(2\pi nx)$$
converges to the constant function 1 almost everywhere on the unit interval $[0, 1]$.

3. Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathfrak{A}, \mu)$ is a measure space. Prove that the set
$$S = \left\{ x \mid \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$$
is a measurable set.

Part II: Choose 2 of the following 3 problems

4. Suppose $(X, \mathfrak{A}, \mu)$ is a measure space and $m(X) = 1$. Suppose f and g are positive measurable functions on X such that $fg \geq 1$. Show that
$$\int_X f \, d\mu(x) \int_X g \, d\mu(x) \geq 1.$$

5. Let $f \in L^1(0, \infty)$ with respect to Lebesgue measure, and suppose that
$$\int_0^\infty |xf(x)| \; dx < \infty.$$
Show that the function
$$g(y) = \int_0^\infty e^{-xy} f(x) \; dx$$
is differentiable at every $y \in (0, \infty)$.

6. Suppose p, q, and r are all greater than 1 and are related by $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Suppose $f \in L^p[0,1]$, $g \in L^q[0,1]$, Show that $fg \in L^r[0,1]$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Part III: Choose 1 of the following 2 problems

7. Suppose $(X, \mathfrak{A}, \mu)$ is a measure space and f is a an integrable function on X. Let $\epsilon > 0$. Show there is an $A \in \mathfrak{A}$, $\mu(A) < \infty$, such that

$$\int_{X|A} |f| \; d\mu < \epsilon.$$

8. Suppose f and g are integrable functions on $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$, respectively, and $F(x,y) = f(x)g(y)$.

   (a) Show that $F(x,y)$ is $(A \otimes B)$-measurable.

   (b) Show that F is integrable of $X \times Y$ and

$$\int_{X \times Y} F \, d(\mu \times \nu) = \int_X f \, d\mu \int_Y g \, d\nu.$$