Instructions: There are five problems on this exam. For each, you have a choice between two problems. Write the number of each problem you work out, and write your name at the top of each page you turn in. You have three hours.

Do exactly one of the following two problems.

1A. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that for each $\epsilon > 0$, one has

$$\mu\{x \in X : |f_n(x)| \geq \epsilon\} \to 0 \quad \text{as} \quad n \to \infty.$$

Prove that there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that for almost every $x \in X$, one has $f_{n_k}(x) \to 0$ as $k \to \infty$.

1B. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_n\}_{n\in\mathbb{N}}$ a sequence of sets in $\mathcal{M}$ such that $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$.

a. If $\mu(X) < \infty$, prove that $\mu(\cap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n)$.

b. Give an example for which $\mu(X) = \infty$ and $\mu(\cap_{n\in\mathbb{N}} E_n) \neq \lim_{n\to\infty} \mu(E_n)$ (prove your assertion).

Do exactly one of the following two problems.

2A. Let f be a real-valued measurable function on a finite measure space $(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x))d\mu \;=\; \mu\{x : f(x) \in \mathbb{Z}\},$$

in which $\mathbb{Z}$ is the set of integers.

2B. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $\{f_n\}_{n=1}^{\infty}$ be in $L^1(X, \mu)$. Suppose that $f_n(x) \to f(x)$ for almost all $x \in X$. Prove that

$$\int |f_n - f|\, d\mu \;\to\; 0 \qquad \Longleftrightarrow \qquad \int |f_n| \;\to\; \int |f|.$$

Do exactly one of the following two problems.

3A. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, y) = x - y$. Let $N \subset \mathbb{R}$ be a Borel set with Lebesgue measure equal to zero. Prove that $f^{-1}(N)$ has Lebesgue measure zero in $\mathbb{R} \times \mathbb{R}$ (that is, with respect to the product of Lebesgue measure in $\mathbb{R}$ with itself).

3B. Let f and g be functions in $L^1(\mathbb{R})$ with respect to Lebesgue measure. Prove the following:

a. The function $H(x, y) := f(x - y)g(y)$ is in $L^1(\mathbb{R}^2)$.

b. The function

$$f * g\,(x) := \int_{\mathbb{R}} f(x - y)\, g(y)\, dy$$

is defined almost everywhere and is an integrable function on $\mathbb{R}$.

c. $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$.

Do exactly one of the following two problems.

4A. Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{M})$. Prove that μ is absolutely continuous with respect to ν if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that, for all $A \in \mathcal{M}$, $\nu(A) < \delta$ implies $\mu(A) < \epsilon$.

4B. Let $k : [0, 1] \to \mathbb{R}$ denote the Cantor ternary function[1], and define a function $F : [0, 1] \to \mathbb{R}$ by

$$F(x) \;=\; \chi_{[1/3,\, 2/3]}(x) \;-\; k(x) \;+\; \sin(\pi x) \;+\; \int_0^x \cos(\pi y)\, dy\, .$$

Answer the following as explicitly as possible.

a. Find the canonical decomposition of F into the difference of increasing functions, $F = F_+ - F_-$, where F_+ and F_- are the positive and negative variations of F. Justify your result.

b. Find the total variation function $T_F(x)$ of $F(x)$.

c. Find the Lebesgue decomposition $\mu_F = \mu_s + \mu_a$ of μ_F into singular (μ_s) and absolutely continuous (μ_a) parts with respect to Lebesgue measure. Here, μ_F is the unique Borel measure on $[0, 1]$ such that $\mu_F((a, b]) = F(b+) - F(a+)$ for $0 \leq a \leq b \leq 1$ (and $\mu_F(\{0\}) = 0$).

d. What is the Radon-Nikodym derivative of μ_a with respect to Lebesgue measure?

e. What can you say about the derivative of F?

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[1]Recall that the Cantor ternary function $k(x)$ is increasing with total variation 1 on $[0, 1]$; $k(x)$ is constant on each of the open intervals that form the complement of the Cantor ternary set; and the Cantor ternary set has Lebesgue measure 0.

Do exactly one of the following two problems.

5A.

a. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that

$$L^q(X, \mathcal{M}, \mu) \subseteq L^p(X, \mathcal{M}, \mu).$$

whenever $1 \leq p \leq q \leq \infty$.

b. Give an example of a measure space $(Y, \mathcal{N}, \nu)$ such that

$$L^p(Y, \mathcal{N}, \nu) \subseteq L^q(Y, \mathcal{N}, \nu)$$

whenever $1 \leq p \leq q \leq \infty$ (prove your assertion).

5B. Suppose that $1 < p \leq q < \infty$ and that $p^{-1} + q^{-1} = 1$, and work with Lebesgue measure on $\mathbb{R}$ and complex-valued functions. Let $(L^p)^*$ denote the dual of L^p; and let $\text{sgn} f$ be defined through the equality $f = |f|\, \text{sgn} f$.
Define a function $\phi : L^p \to L^q$ by

$$\phi(f) \, = \, |f|^{p-1} \text{sgn}\, f$$

and another function $: L^q \to (L^p)^*$ by

$$(g)\, f \, = \int gf \qquad \forall \quad f \in L^p \, .$$

Prove that $\circ\, \phi : L^p \to (L^p)^*$ is a bijection that is nonlinear and unbounded whenever $p \neq 2$.